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ON THE EQUATION $\left(3xy^2 - x^3\right)^2 + \left(3x^2y - y^3\right)^2 = 8z^{12}$

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ABSTRACT

We presents non zero solutions of the 12 th degree non-homogeneous Diophantine equation in three unknowns represented by $\left(3xy^2 - x^3\right)^2 + \left(3x^2y - y^3\right)^2 = 8z^{12}$. *A few interesting properties among the solutions are exhibited.*

Keywords: Higher degree equation with three unknowns, Integral solutions, polygonal numbers, star numbers.

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Introduction

Diophantine equations have an unlimited field for research by reason of their variety. These equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians [1-6]. The problem of finding all integer solutions of a Diophantine equation with three or more variables and degree at least three, in general, presents a good deal of difficulties. There is vast general theory of homogeneous quadratic equations with three variables. Cubic equations in two variables fall into the theory of elliptic curves which is a very developed theory but still an important topic of current research [7,8]. A lot is known about equations in two variables in higher degrees. For equations with more than three variables and degree at least three very little is known. In [9,10] a few higher order equations are considered for integral solutions. In this communication a twelfth degree diophantine equation with three variables represented by is considered and in particular a few interesting relations among the solutions are presented.

Notations

$Hex_n = Hexagonal\ number\ of\ rank\ n = 2n^2 - n$

$Star_n = Star\ number\ of\ rank\ n = 6n(n-1)+1$

$CH_n = Centered\ hexagonal\ number\ of\ rank\ n = 3n^2 - 3n + 1$

$SO_n = Stella$ Octangula $number\ of\ rank\ n = n(2n^2 - 1)$

$\text{Pro}_n = \text{Pronic}\ number\ of\ rank\ n = n(n+1)$

$To_r = Truncated$ octahedral number $of\ rank\ r = 16r^3 - 33r^2 + 24r - 6$

Method of Analysis

The equation under consideration is

$$\left(3xy^2 - x^3\right)^2 + \left(3x^2y - y^3\right)^2 = 8z^{12} \quad (1)$$

By applying the linear transformations

$$x = m + n\ and\ y = m - n \quad (2)$$

in equation (1), it reduced to

$$m^2 + n^2 = z^4 \quad (3)$$

Which is similar to well known Pythagorean equation

$$m^2 + n^2 = \left(z^2\right)^2 \quad (4)$$

Thus the solution of equation (4) are taken as

$$m = p^2 - q^2,\ n = 2pq\ and\ z^2 = p^2 + q^2 \quad (5)$$

In equation (5), the expression $z^2 = p^2 + q^2$ is also a Pythagorean equation.

Therefore, $p = r^2 - s^2,\ \text{q} = 2rs\ and\ z = r^2 + s^2$ (6)

Applying equation (6) in (5), and thenin (2) the solution of equation (1) is,

$$x(r,s) = \left(r^2 - s^2\right)^2 - 4r^2s^2 + 4rs\left(r^2 - s^2\right)$$

$$y(r,s) = \left(r^2 - s^2\right)^2 - 4r^2s^2 - 4rs\left(r^2 - s^2\right)$$

$$z(r,s) = \left(r^2 + s^2\right)$$

Numerical examples

r	s	x	y	z
1	2	-23	25	5
2	3	-239	4	13
3	1	124	-68	10
3	4	-863	-191	25
4	1	401	-79	17

Proposition 1

The following expressions forms Nasty number

$i)\quad 3\left[x(r,s)+y(r,s)+8r^2s^2\right]$

$ii)\quad 180\left(r^4+s^4\right)-60\left[x(r,s)+y(r,s)+z^2(r,s)\right]$

Solution:

Clearly, $x(r,s)+y(r,s)=2\left(r^2-s^2\right)^2-8r^2s^2$

Therefore, $3\left[x(r,s)+y(r,s)+8r^2s^2\right]$ forms a nasty number. Which is i)

ii) $x(r,s)+y(r,s)+z^2(r,s)=3\left(r^4+s^4\right)-10r^2s^2$

so, $30\left(r^4+s^4\right)-10\left[x(r,s)+y(r,s)+z^2(r,s)\right]=\left(100r^2s^2\right)$

$180\left(r^4+s^4\right)-60\left[x(r,s)+y(r,s)+z^2(r,s)\right]$ forms a nasty number.

Hence the proof.

Proposition 2

$x(r,1)-y(r,1)+z(r,1)+6r+4=To_r+11Ch_r-4So_r+\text{Pro}_r$

Solution:

Cleary, $x(r,s)-y(r,s)+z(r,s)=8r^3s-8rs^3+r^2+s^2$

$$x(r,1)-y(r,1)+z(r,1)=8r^3-8r+r^2+1$$

$$= To_r - 8r^3 - 32r + 34r^2 + 7$$

$$= To_r + 11Ch_r - 8r^3 + 4r + r^2 - 5r + 4$$

$$= To_r + 11Ch_r - 4So_r + \text{Pro}_r - 6r + 4$$

Hence, $x(r,1) - y(r,1) + z(r,1) + 6r + 4 = To_r + 11Ch_r - 4So_r + \text{Pro}_r$

Proposition 3

$x(r,1)\,y(r,1) = \text{Difference of two squares}$

Solution:

Clearly, $x(r,1)\,y(r,1) = \left[\left(r^2 - s^2\right)^2 - 4r^2s^2\right]^2 - \left[4rs\left(r^2 - s^2\right)\right]^2$

Therefore, $x(r,1)\,y(r,1)$ is difference of two squares.

Proposition 4

$x(1,s) + y(1,s) + z(1,s) = Hex_{s^2} - 2Star_s - 12s + 6$

Solution:

$x(1,s) + y(1,s) + z(1,s) = 3 + 2s^4 - 13s^2$

$$= 3 + Hex_{s^2} - 12s^2$$

$$= 6 - 12s + Hex_{s^2} - 2Star_s$$

Hence the proof.

Proposition 4

$x(r,s) + y(r,s) + z^2(r,s) + 10r^2s^2$ is three times sum of quartic integer.

Solution:

Clearly, $y(r,s)+z^2(r,s)=2r^4+2s^4-4r^2s^2-4r^3s+4rs^3$

$x(r,s)+y(r,s)+z^2(r,s)=3r^4+3s^4-10r^2s^2$

Hence, $x(r,s)+y(r,s)+z^2(r,s)-10r^2s^2$ = 3(sum of two quartic integers)

Proposition 5

$30(r^4+s^4)-10[x(r,s)+y(r,s)+z^2(r,s)]$

Solution:

From proposition 1, we have

$x(r,s)+y(r,s)+z^2(r,s)=3(r^4+s^4)-10r^2s^2$

therefore, $3(r^4+s^4)-10[x(r,s)+y(r,s)+z^2(r,s)]$ is a perfect square.

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