**International Research Journal of Natural and Applied Sciences**
ISSN: (2349-4077)
**Impact Factor**- 5.46, Volume 4, Issue 7, July 2017
**Website**- www.aarf.asia, **Email** : editor@aarf.asia , editoraarf@gmail.com

# OPTIMAL FUND ALLOCATION FROM CERTAIN INVESTMENT PORTFOLIO USING BACKWARD DYNAMIC PROGRAMMING RECURSIONS

**[1]Ukwu Chukwunenye; [2]Manjel Danladi & [3]Kutchin Stephen**
[1,2,3]Department of Mathematics, University of Jos, P.M.B 2084, Jos, Plateau State, Nigeria.

**ABSTRACT**

*This research article investigated the problem of fund allocations from certain investment portfolio, formulated and proved an appropriate theorem on the accumulated funds at the end of each investment year and then appropriated the results to obtain the optimal investment strategies using backward dynamic programming recursive approach. In the sequel, the work provided illustrative examples which demonstrated the inherent tedious and prohibitive manual computations associated with dynamic programming computations and the imperative of digital generation of optimal fund allocation prescriptions, in subsequent papers.*

**Keywords:** Dynamic programming, Computations, Optimal investment, Fund allocations, Investment portfolio.

## INTRODUCTION

Today's investors are faced with several challenges as they look for comfortable ways to earn higher returns on their investments above the current certificate of deposit (CD) and interest rates. One of the challenges is the efficient allocation of funds, which is one of the most important functions of the financial management in modern times, Gupta & Hira [1].

Ukwu [2] obtained the proof of the optimal investment strategy and corresponding rewards for a class probabilistic stationary investment problems, using backward dynamic programming recursive approach. In the sequel, the article formulated nontrivial extensions of the results to a larger dynamic class for practical and realistic considerations. The recursions were based on conditional probabilities and the proofs were achieved by deft deployment of probability axioms, set-theoretic facts, optimization and inductive principles. The extensions reflected and demonstrated consistency with the base results.

Taha [3] investigated the optimal investment policy for a portfolio of two banks in a certain class of deterministic investment problems using deterministic dynamic programming recursions. Unfortunately the related issue of computational feasibility is yet to be addressed. Dynamic programming iterations are computationally intractable and doomed to failure for practical purposes, especially for large scale applications.

The main purpose of this research is to extend the optimal policy prescriptions for fund allocation to the same class of investment problems involving an arbitrary number of banks, from the formulation and proof of an appropriate theorem on cash-flow pattern and translation of the latter to dynamic programming (D.P.) recursions with illustrative solutions for problem instances. The findings of this study will be of great benefits to the financial sector of the economy, especially to investors and bank managers.

This study will also provide illustrative examples which also demonstrate the computational tedium inherent in D.P. solution processand other constraints militating against the pursuit of sensitivity analyses on the pertinent data. Manual computational activities could hardly be contemplated for large-scale problems.

## 2. THEORETICAL UNDERPINNING

### 2.1 Compound Interest Computations

The Total Accumulated Value (TAV), is the principal sum plus compounded interest and is given by the formula:

$$\text{TAV} = P\left(1+\frac{i}{n}\right)^{nt},$$

where $P$ is the principal sum; $i$ is the nominal interest rate; $n$ is the annual compounding frequency and $t$ is the planning horizon length the interest is applied.

## 2.2 Optimal Investment Policies for Two Banks

Given: $P_1, P_2, ..., P_n$ – Amounts to be invested at the start of each of the next $n$ years.

$r_1, r_2$ – Nominal interest rates for banks 1 and 2, respectively, for annual compounding.

$q_{i,1}, q_{i,2}$ – Bonuses paid at the end of year $i$ in which the investments are made in the two banks

$x_i$ – Amount of capital available for investment at the start of year $i$.

$S_i$ – Accumulated sum at the end of year $i$ given that $I_{i,1}$, $I_{i,2}$ are the investments made in the two banks at the beginning of the year $i$

$$x_1 = I_{1,1} + I_{1,2}, x_2 = I_{2,1} + I_{2,2}, \ x_i = \sum_{j=1}^{2} I_{i,j}$$

Letting $\alpha_{n,j} = \prod_{i=1}^{n} (1 + r_{n,j}), j = 1, 2,$ the problem can be stated as follows:

$$\text{Maximize z} = \sum_{i=1}^{n} S_i,$$

where

$$s_i = \sum_{j=1}^{2} I_{i,j} \alpha_{i,j}^{n+1-i}; i \in \{1, 2, ...n-1\}$$

(incorporating bonuses and fresh deposits)

$$s_n = \sum_{j=1}^{2} I_{n,j} \alpha_{n,j} + \sum_{j=1}^{2} q_{n,j} I_{n,j} = \sum_{j=1}^{2} I_{n,j} \left[ \alpha_{n,j} + q_{n,j} \right]$$

Define

$f_i(x_i)$ = optimal value of investments for years $i, i+1$, ..., and $n$ given $x_i$

The last statement arises from the fact that the bonuses for year $n$ are part of the final accumulated sum of the money from the investments.

$$x_1 = P_1 \ , \ x_i = P_i + \sum_{j=1}^{2} q_{i-1,j} I_{i-1,j}, i \in \{1, 2, ...n\}$$

The backward DP recursion equation is thus given as:

$$f_i(x_i) = \max_{0 \le I_i \le x_i} \{s_i + f_{i+1}(x_{i+1})\}, i = 1, 2, ..., n-1$$

$$f_{n+1}(x_{n+1}) = 0.$$

As given previously, $x_{i+1}$ is defined in terms of $x_i$

In this research, the above results will be extended to an arbitrary number of banks.

## 2.3 Hypotheses of the problem

$k$ : Number of banks to be invested in

$P_i$ : Fixed component of the amount for investment in all $k$ banks at the start of year $i$, $i \in \{1, 2, ...n\}$

$r_{i,j}$ : Nominal annual interest rate offered by bank $j$ in year $i$ , $j \in \{1, 2, ..., k\}$

$q_{i,1}, q_{i,2}, ...q_{i,k}$ : Bonuses percentages paid at the end of year $i$ in which investment is made in all $k$ banks.

$x_i$ : Variable amount available for investment in all $k$ banks in year $i$(includes $P_i$ )

$I_{i,j}$ : Actual amount invested in bank $j$ at the beginning of year $i$

$s_i$ : Accumulated sum at the end of year $i$ given that $I_{i,j}$ is the investment made in bank $j$ at the

beginning of year $i$ , $j \in \{1, 2, ..., k\}; i \in \{1, 2, ..., n\}$

## 2.4 Problem Statement

$$\text{Maximize } z = \sum_{i=1}^{n} s_i$$

$$\text{s.t.} \quad s_i = \sum_{j=1}^{k} I_{i,j} \alpha_{i,j}^{n+1-i}, i \in \{1, 2, ...n-1\}$$

$$s_n = \sum_{j=1}^{k} I_{n,j} \alpha_{n,j} + \sum_{j=1}^{k} q_{n,j} I_{n,j} = \sum_{j=1}^{k} I_{n,j} \left[ \alpha_{n,j} + q_{n,j} \right]$$

$k = 2$ yields the following:

$$s_i = I_{i,1} \alpha_{i,1}^{n+1-i} + I_{n,2} \alpha_{i,2}^{n+1-i};\ s_n = I_{n,i} (\alpha_{i,1} + q_{n,1}) + I_{n,2} (\alpha_{i,2} + q_{n,2})$$

The relevant expressions will now be extended to an arbitrary numberof banks.

## 3. RESULTS AND DISCUSSION

### 3.1 Theorem on Available and Accumulated Funds

The general expressions for $s_i, s_n$, and $x_i$ are given as follows:

$$s_i = \sum_{j=1}^{k} I_{i,j}\alpha_{i,j}^{n+1-i}; i \in \{1,2,...,n-1\}; s_n = \sum_{j=1}^{k} (\alpha_{n,j} + q_{n,j})\, I_{n,j}; x_i = p_i + \sum_{j=1}^{k} q_{i-1,j} I_{i-1j}; i \in \{1,2,...,n\}$$

**Proof**

The proof must justify the expressions for $x_i, s_i, s_n, Z$ as well as the DP recursions. The principle of

mathematical induction must be applied on the number of banks. First, the notations are modified as follows:

$s_i(k)$ in place of $S_i$; $s_n(k)$ in place of $s_n$; $x_i(k)$ in place of $x_i$.

Set

$$q_{i,j} = 0, \text{ for } i \le 0, \forall j \in \{1,2,\cdots,k\}.$$

Thus, we wish to prove the following:

$$s_i(k) = \sum_{j=1}^{k} I_{i,j}\alpha_{i,j}^{n+1-i},\ i \in \{1,2,...,n-1\} \quad (1)$$

$$s_n(k) = \sum_{j=1}^{k} (\alpha_{n,j} + q_{n,j})\, I_{n,j} \quad (2)$$

$$x_i(k) = P_i + \sum_{j=1}^{k} q_{i-1,j} I_{i-1,j};\ i \in \{1,2,...,n\} \quad (3)$$

#### 3.1.1 Proof of expression (1)

Let $k = 1$. Then for $i = 2$

$$s_1(1) = I_{1,1}\alpha_{1,1}^{n},\ s_n(1) = I_{n,1}\alpha_{n,1}, s_1(2) = I_{1,1}\alpha_{1,1}^{n} + I_{1,2}\alpha_{1,2}^{n};\ i = 2,$$
$$k = 2 \Rightarrow s_2(2) = I_{2,1}\alpha_{2,1}^{n-1} + I_{2,2}\alpha_{2,2}^{n-1},$$

since there are $n+1-i$ years from the beginning of year $i$ to the end of year $n$. The investment $I_{i,j}$ made at the beginning of year $i$ in bank $j$ has the future value of $I_{i,j}\alpha_{i,j}^{n+1-i}$ at the end of year $n$. As can be seen from the results already established, the theorem is valid for $k \in \{1,2\}$.

The rest of the proof is by the principle of mathematical induction.

Assume that the theorem is valid for $k \in \{1, 2, \cdots, m\}$, for some positive integer $m \geq 3$. The theorem must be proved to be valid for $k = m+1$.

By the induction hypothesis,

$$s_i(m) = \sum_{j=1}^{m} I_{i,j}\alpha_{i,j}^{n+1-i}; i \in \{1, 2, ..., n-1\}$$

The numbering of the banks is arbitrary. So without loss of generality, let the first $m$ banks be numbered 2, 3,..., $m+1$.

Set

$$N_m = \{2, 3..., m+1\}.$$

Then

$$s_i(m+1) = \sum_{j=2}^{m+1} I_{i,j}\alpha_{i,j}^{n+1-i} + I_{i,1}\alpha_{i,1}^{n+1-i} = s_i(m) + s_i(1),$$

for $i \in \{1, 2, ..., n-1\}$. Observe that $1 \notin N_m$.

Therefore

$$s_i(m+1) = \sum_{j=1}^{m+1} I_{i,j}\alpha_{i,j}^{n+1-i},$$

$\Rightarrow$ the formula is valid for $m+1$ banks and hence valid for an arbitrary number of banks; that is to say that the formula

$$s_i(k) = \sum_{j=1}^{k} I_{i,j}\alpha_{i,j}^{n+1-i}$$

is valid for an arbitrary number of banks, $k \in \{1, 2 \ldots\}$

### 3.1.2 Proof of expression (2)

The proof is by mathematical induction on $k$. Let $k = 1$ and $i = n$. Then

$$s_n(1) = (\alpha_{n,1} + q_{n,1})I_{n,1},\ s_n(2) = (\alpha_{n,1} + q_{n,1})I_{n,1} + (\alpha_{n,2} + q_{n,2})I_{n,2}$$

From the above results, the theorem is valid for $k \in \{1, 2\}$.

Assume that the theorem is valid for $k \in \{3, \ldots m\}$, for some positive integer $m$. Then the theorem must be proved for $k = m+1$.

Thus,

$$s_n(m) = \sum_{j=1}^{m} (\alpha_{n,j} + q_{n,j})I_{n,j} \text{ is valid for } m \geq 3.$$

Without loss of generality, let the first $m$ banks be numbered {2, 3,..., $m+1$}


A Monthly Double-Blind Peer Reviewed Refereed Open Access International e-Journal - Included in the International Serial Directories.

$$s_n(m+1) = s_n(m) + s_n(1); 1 \notin N_m$$

$$s_n(m+1) = \sum_{j=2}^{m+1} (\alpha_{n,j} + q_{n,j}) I_{n,j} + (\alpha_{n,1} + q_{n,1}) I_{n,j}$$

Therefore, $$s_n(m+1) = \sum_{j=2}^{m+1} (\alpha_{n,j} + q_{n,j}) I_{n,j},$$

establishing the validity of the formula for $k = m+1$, hence

$$s_n(k) = \sum_{j=1}^{k} (\alpha_{n,j} + q_{n,j}) I_{n,j}$$

### 3.1.3 Proof of expression (3)

We proceed with the proof of expression (3) by the principle of mathematical induction on $k$.

$$k = 1, i = 1 \Rightarrow x_1(1) = p_1 + q_{0,1} + I_{0,1} \Rightarrow x_1(1) = P_1$$

$$i = 1, k = 2 \Rightarrow x_1(2) = P_1 + q_{0,1} I_{0,1} + q_{0,2} I_{0,1} \Rightarrow x_1(2) = P_1$$

$$i = 2,\ k = 2 \Rightarrow x_2(2) = P_2 + q_{1,1} I_{1,1} + q_{1,2} I_{1,2}$$

It can be seen, from the above results, that the variable amount available for component of the amount for investment at the start of year 1 is equal to the fixed component of the amount for investment in bank 1and bank 2, since no bonus has been paid, neither has any interest been generated.

Assume that the theorem is valid for $k \in \{3, \ldots, m\}$, for some positive integer $m \geq 3$. By the induction hypothesis, the expression

$$x_i(m) = P_i + \sum_{j=1}^{m} q_{i-1,j} I_{i-i,j;}\ i \in \{1, 2, \ldots n\}$$

is valid. The theorem must be proved to be valid for $k = m+1$. The numbering of the banks is arbitrary. So without loss of generality, let the first $m$ banks be numbered $2, 3, \ldots, m, m+1$

Set $N_m = \{2, 3, \ldots, m+1\}$; then $1 \notin N_m$. Furthermore,

$$x_i(m+1) = x_i(m) + x_i(1) = P_i + \sum_{j=2}^{m+1} q_{i-1j} I_{i-1,j} + P_i + q_{i-1,1} I_{i-1,1} \Rightarrow x_i(m+1) = P_i + \sum_{j=1}^{m+1} q_{i-1,j} I_{i-1,j}$$

Therefore the formula is valid for $m+1$ banks and hence valid for an arbitrary number of banks:

$$x_i(k) = P_i + \sum_{j=1}^{k+1} q_{i-1,j} I_{i-1,j} \quad k \in \{1, 2...\}$$

## 3.2 Application Problems

### 3.2.1 Application to 3-Bank-4-Year Problem

Suppose that you want to invest \$20,000.00 now and \$10,000.00 at the start of year 2, 3 and 4. The interest rate offered by banks 1 is 5% compounded annually, and the bonuses over the next 4 years are 2%, 1.7%, 2.5% and 3.2% respectively. The annual interest rate offered by bank 2 is lower by 0.015 than that of bank 2, but its bonus is higher by 0.02. The annual interest rate offered by bank 3 is 3.3% and its bonuses over the next 4 years are 1.2% 1.65%, 1.5% and 2.1% respectively. The objective is to maximize the accumulated capital at the end of 4 years.

Define

$f_i\left(x_i\right) =$ Optimal value of the investments for years $i, i+1, ...,$ and $n$ , given $x_i$

The backward DP recursive equation are thus given as

$$f_i(x_i) = \max_{\substack{0 \le I_{i,j} \le x_i \\ j \in \{1,2,3\}}} \{s_i + f_{i+1}(x_{i+1})\}, i = 1, 2..., n-1$$

**Note the following**

$f_i(x_i) = \max\limits_{0 \le I_{i,j} \le x_i; j \in \{1,2,\cdots,k\}} g_i\left(I_{i,1}, I_{i,2}, \cdots I_{i,k}\right)$, for some function, $g$.

Suppose the maximum of $g_i$ is attained at $I_{i,j^*}$, for some $j^* \in \{1, 2, \cdots, k\}$, then

$I_{i,j^*} = \arg\max\ g_i\left(I_{i,1}, I_{i,2}, \cdots I_{i,k}\right)$; this is the optimal investment decision in year $i$.

$I_{i,j_i^*} = \arg\max\ g_i\left(I_{i,1}, I_{i,2}, \cdots I_{i,k}\right)$ is an alternative representation.

Using the notations introduced previously leads to the interpretation:

$$P_1 = \$20,000:00, P_2 = P_3 = \$10,000:00, \alpha_1 = (1+0.05) = 1.05, \alpha_2 = (1+0.35) = 1.035$$
$$q_{1,1} = 0.02,\ q_{2,1} = 0.017,\ q_{3,1} = 0.025,\ q_{4,1} = 0.032; q_{1,2} = 0.04,\ q_{2,2} = 0.037,\ q_{3,2} = 0.045,$$
$$q_{4,2} = 0.052;\ q_{1,3} = 0.012,\ q_{2,3} = 0.0165,\ q_{3,3} = 0.015,\ q_{3,4} = 0.021$$

The calculations are in thousands of dollars.

**Stage 4 Computations**

$$f_4(x_4) = \max_{\substack{0 \le I_{4j} \le x_4 \\ j \in \{1,2,3\}}} \{s_4 + f_5(x_5)\},\ f_5\left(x_5\right) = 0 \Rightarrow f_4(x_4) = \max_{\substack{0 \le I_{4,j} \le x_4 \\ j \in \{1,2,3\}}} \{s_4\}$$

where

$$s_4 = (\alpha_{4,1} + q_{4,1})I_{4,1} + (\alpha_{4,2} + q_{4,2})I_{4,2} + (\alpha_{4,3} + q_{4,3})I_{4,3}$$
$$= (1.05 + 0.032)\, I_{4,1} + (1.03 + 0.052)I_{4,2} + (1.033 + 0.021)$$
$$= 1.082 I_{4,1} + 1.087 I_{4,2} + 1.054 I_{4,3}$$

**Table 1: Summary of Optimal Policy and Return for Year 4**

| **Optimum solution** | | | **Optimal decision** |
|---|---|---|---|
| State | $j^*$ | $f_4(x_4)$ | $I_4^*$ |
| $x_4$ | 2 | $1.087\, x_4$ | $I_{42}^* = x_4$ |

## Stage 3 Computations

$$f_3(x_3) = \max_{\substack{0 \le I_{3,j} \le x_3 \\ j \in \{1,2,3\}}} \{s_3 + f_4(x_4)\},$$

where

$$s_3 = \alpha_{3,1}^2 I_{3,1} + \alpha_{3,2}^2 I_{3,2} + \alpha_{3,3}^2 I_{3,3} = (1.05)^2 I_{3,1} + (1.035)^2 I_{3,2} + (1.033)^2 I_{3,3}$$
$$= 1.1025 I_{3,1} + 1.071225 I_{3,2} + 1.067089 I_{3,3}$$
$$x_4 = P_4 + q_{3,1} I_{3,1} + q_{3,2} I_{3,2} + q_{3,3} I_{3,3} = 10{,}000 + 0.025\, I_{3,1} + 0.045 I_{3,2} + 0.015 I_{3,3}$$

$$\Rightarrow f_3(x_3) = \max_{\substack{0 \le I_{3,j} \le x_3 \\ j \in \{1,2,3\}}} \{1.1025 I_{3,1} + 1.071225 I_{3,2} + 1.067089 I_{3,3} + 1.087 x_4\}$$
$$= \max_{\substack{0 \le I_{3,j} \le x_3 \\ j \in \{1,2,3\}}} \{1.1025 I_{3,1} + 1.071225 I_{3,2} + 1.067089 I_{3,3} + 1.087(10,000 + 0.025 I_{3,1} + 0.045 I_{3,2} + 0.015 I_{3,3})\}$$
$$= \max_{\substack{0 \le I_{3j} \le x_3 \\ j \in \{1,2,3\}}} \{1.129675 I_{3,1} + 1.12014 I_{3,2} + 1.083394 I_{3,3} + 10870\}$$

**Table 2: Summary of Optimal Policy and Return for Years 3 and 4**

| **Optimum Solution** | | | **Optimal Decision** |
|---|---|---|---|
| State | $j^*$ | $f_3(x_3)$ | $I_{3,j^*}$ |
| $x_3$ | 1 | $10870 + 1.12967\, x_3$ | $I_{3,1} = x_3$ |

## Stage 2 Computations

$$f_2(x_2) = \max_{\substack{0 \le I_{2j} \le x_2 \\ j \in \{1,2,3\}}} \{s_2 + f_3(x_3)\},$$

where

$$s_2 = \alpha_{2,1}^3 I_{2,1} + \alpha_{2,2}^3 I_{2,2} + \alpha_{2,3}^3 I_{2,3} = (1.05)^3 I_{21} + (1.035)^3 I_{22} + (1.033)^3 I_{23}$$

$$= 1.157625\mathrm{I}_{2,1} + 1.10871788 I_{2,2} + 1.1.0230294 I_{2,3}$$

$$x_3 = p_3 + \mathrm{q}_{2,1}\mathrm{I}_{2,1} + \mathrm{q}_{2,2}\mathrm{I}_{2,2} + \mathrm{q}_{2,3} I_{2,3} = 10{,}000 + 0.017\mathrm{I}_{2,1} + 0.037 I_{2,2} + 0.0615 I_{2,3},$$

thus,

$$f_2(x_2) = \max_{\substack{0 \le I_{2j} \le x_2 \\ j \in \{1,2,3\}}} \{1.157625 I_{2,1} + 1.10871788 I_{2,2} + 1.10230294 I_{2,3} + 10870 + 1.129675 x_3)$$

$$= \max_{\substack{0 \le I_{2j} \le x_2 \\ j \in \{1,2,3\}}} \{1.157625 I_{2,1} + 1.10871788 I_{2,2} + 1.10230294 I_{2,3} + 10870$$

$$+ 1.129675(10,000 + 0.017 I_{2,1} + 0.037 I_{2,2} + 0.0165 I_{2,3})\}$$

$$= \max_{\substack{0 \le I_{2j} \le x_2 \\ j \in \{1,2,3\}}} \{1.17682948 I_{2,1} + 1.15051586 I_{2,2} + 1.12094258 I_{2,3} + 22166.75\}$$

**Table 3: Summary of Optimal Policy and Return for Stage 2**

| **Optimum Solution** | | | **Optimal Decision** |
|---|---|---|---|
| State | $j^*$ | $f_2(x_2)$ | $I_{2,j^*}$ |
| $x_2$ | 1 | 2,22166.75 +1.17682948 $x_2$ | $I_{2,1} = x_2$ |

**Stage 1 computations**

$$f_1(x_1) = \max_{\substack{0 \le I_{1,j} \le x_1 \\ j \in \{1,2,3\}}} \{s_1 + f_2(x_2)\},$$

where

$$s_1 = \alpha_{1,1}^4 I_{1,1} + \alpha_{1,2}^4 I_{1,2} + \alpha_{1,3}^4 I_{1,3} = (1.05)^4 I_{1,1} + (1.035)^4 I_{1,2} + (1.033)^4 I_{1,3}$$

$$= 1.21550625\mathrm{I}_{1,1} + 1.147523 I_{1,2} + 1.13867893 I_{1,3}$$

$$x_2 = p_2 + q_{1,1} I_{1,1} + q_{1,2} I_{1,2} + q_{1,3} I_{1,3} = 10,000 + 0.02 I_{1,1} + 0.04 I_{1,2} + 0.012 I_{1,3}$$

$$\Rightarrow f_1(x_1) = \max_{\substack{0 \le I_{1,j} \le x_1 \\ j \in \{1,2,3\}}} \{1.21550625 I_{1,1} + 1.147523 I_{1,2} + 1.13867893 I_{1,3} + 22166.75 + 1.17682948 x_2\}$$

$$= \max_{\substack{0 \le I_{1,j} \le x_1 \\ j \in \{1,2,3\}}} \{1.21550625 I_{1,1} + 1.147523 I_{1,2} + 1.13867893 I_{1,3} + 22166.75 + 1.17682948(10,000 + 0.02 I_{1,1} + 0.012 I_{13})\}$$

$$= \max_{\substack{0 \le I_{1j} \le x_1 \\ j \in \{1,2,3\}}} \{1.23904284 I_{1,1} + 1.19459618 I_{1,2} + 1.15280088 I_{1,3} + 33935.0448\}$$

**Table 4: Summary of Optimal Policy and Return for Stage 1**

| Optimal Solution | | | Optimal Decision |
|---|---|---|---|
| State | $j^*$ | $f_1(x_1)$ | $I_{1,j^*}$ |
| $x_1 = \$20,000$ | 1 | 33935.0448+1.2390484 $x_1$ | $I_{11} = x_1$ |

Working backwards and noting that $I_{1,1} = \$20,000 = x_1$ , $I_{2,1} = x_2, I_{4,2} = x_4$

$I_{3,1} = x_3, x_2 = 10,000 + 0.02 I_{1,1} + 0.04 I_{1,2} + 0.012 I_{1,3}$ =10,000 + 0.023 (20,000) + 0.04 (0) + 0.012 (0)

= \$10,400

$x_3 = 10,000 + 0.017 I_{2,1} + 0.037 I_{2,2} + 0.0165 I_{2,3}$ = 10,000 + 0.017 ( 10400) + 0.037 (0) +0.0165 (0)

= \$10,176.8

$x_4 = \$10,000 + 0.025 I_{3,1} + 0.045 I_{3,2} + 0.0156 I_{3,3}$ = 10,000 +0.025 (10176.8) + 0.045 (0) + 0.015(0)

= \$10,254.42

$s_1 = 1.21550625 I_{1,1} + 1.147523 I_{1,3} + 1.13867893 I_{1,3}$

= 1.21550625 (20,000) + 1.147523 (0) + 1.13867893 (0) =\$24310.125

$s_2 = 1.157625 I_{2,1} + 1.10871788 I_{2,2} + 1.10230294 I_{2,3}$

= 1.157625 (10400) +1.10871788 (0) + 1.10230294 (0) = \$12039.3

$s_3 = 1.1025 I_{3,1} + 1.071225 I_{3,2} + 1.067089 I_{3,3}$

=1.1025 (10176.8) + 1.071225 (0) + 1.067089 (0) = \$11219.922

$s_4 = 1.082 I_{4,1} + 1.087 I_{4,2} + 1.054 I_{4,3}$

=1.082 (0) + 1.087 (10254.42) + 1.054 (0) =\$11,146.5545

**The Optimal solution is thus summarized as follows:**

**Table 5: Summary of the Optimal Policies and Returns for Years 1 to 4**

| Year | Optimal solution | Decision | Accumulation |
|---|---|---|---|
| 1 | $j_1^* = I_{1,1} = x_1$ | Invest $x_1$= \$20,000 in Bank 1 | $s_1 = \$24,310.13$ |
| 2 | $j_2^* = I_{21} = x_2$ | Invest $x_2$ =\$10400 in Bank 1 | $s_2 = \$12,039.30$ |
| 3. | $j_3^* = I_{3,1} = x_3$ | Invest $x_3$ =\$10176.8 in Bank 1 | $s_3 = \$11219.92$ |
| 4. | $j_4^* = I_{4,2} = x_4$ | Invest $x_4$ =\$10254.42 in Bank 1 | $s_4 = \$1146.55$ |

**TOTAL ACCUMULATION =\$ 58,715.90**

### 3.2.2 Application to 4-Bank-5-Year Investment Problem

Given the amount \$15,000 to be invested now and \$ 8,000, \$ 12,000, \$ 5,000 and \$13, 000 at the start of years 2,3 4 and 5, Table 2 furnishes the investment funds, pertinent nominal annual interest rates and end-of-year bonuses offered by a portfolio of three banks for a period of five successive years:

**Table 6: Pertinent Data for Application problem 3.2.2**

| Bank | | | | | | Bank | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
| Year | $P$ | Nominal Annual Interest Rates | | | | End-of-Year Bonuses | | | |
| 1 | 15,000 | 0.012 | 0.0132 | 0.0136 | 0.07 | 0.02 | 0.013 | 0.015 | 0.022 |
| 2 | 8,000 | 0.012 | 0.0132 | 0.0136 | 0.07 | 0.018 | 0.011 | 0.030 | 0.020 |
| 3 | 12,000 | 0.012 | 0.0132 | 0.0136 | 0.07 | 0.03 | 0.023 | 0.027 | 0.083 |
| 4 | 5,000 | 0.012 | 0.0132 | 0.0136 | 0.07 | 0.032 | 0.025 | 0.0175 | 0.050 |
| 5 | 13,000 | 0.012 | 0.0132 | 0.0136 | 0.07 | 0.015 | 0.08 | 0.042 | 0.037 |

Devise the optimal investment strategies.

**Solution**

Define

$f_i(x_i)$ = optimal value of the investments for years $i, i+1, \ldots,$ and $n$ , given $x_i$

The objective is to maximize the accumulated capital at the end of 5 years.

The backward DP recursive equations are given as:

$$f_i(x_i) = \max_{\substack{0 \le I_{i,j} \le x_i \\ j \in \{1,2,3,4\}}} \{s_i + f_{i+1}(x_{i+1})\}, i = 1, 2, \ldots, n-1$$

$$f_{n+1}(x_{n+1}) = 0$$

Using the notations introduced the investment parameters are as follows:

$P_1 = \$15,000, P_2 = \$8,000, P_3 = \$12,000, P_4 = \$5,000$ and $P_5 = \$13,000, \alpha_i = \alpha_{i,j}, \forall\, i \in \{1,2,3,4,5\},$ for each $j \in \{1,2,3\}$, where $\alpha_1 = (1+0.012) = 1.012, \alpha_2 = (1+0.132) = 1.132, \alpha_3 = (1+0.136) = 1.136,$ $\alpha_4 = (1+0.07) = 1.07;$

$q_{1,1} = 0.02,\ q_{2,1} = 0.018,\ q_{3,1} = 0.03,\ q_{4,1} = 0.032,\ q_{5,1} = 0.015$

$q_{1,2} = 0.013\ q_{2,2} = 0.011,\ q_{3,2} = 0.023\ q_{4,2} = 0.025,\ q_{5,2} = 0.08$

$q_{1,3} = 0.015,\ q_{2,3} = 0.03,\ q_{3,3} = 0.027,\ q_{4,3} = 0.0175,\ q_{5,3} = 0.042$

$q_{1,4} = 0.022,\ q_{2,4} = 0.02,\ q_{3,4} = 0.083,\ q_{4,4} = 0.05,\ q_{5,4} = 0.037$

**Stage 5 Computations**

$$f_5(x_5) = \max_{\substack{0 \le I_{5,j} \le x_5 \\ j \in \{1,2,3,4\}}} \{s_5 + f_6(x_6) = \max_{\substack{0 \le I_{5,j} \le x_5 \\ j \in \{1,2,3,4\}}} \{s_5\}$$

where

$$s_5 = (\alpha_1 + q_{5,1})I_{5,1} + (\alpha_2 + q_{5,2})I_{5,2}(\alpha_3 + q_{5,3})I_{5,3} + (\alpha_4 + q_{5,4})I_{5,4}$$
$$= 1.027\mathrm{I}_{5,1} + 1.212 I_{5,2} + 1.178 I_{5,3} + 1.107 I_{5,4}$$

**Table 7: Summary of the Optimal Investment Policy and Return for Stage 5**

| **Optimal solution** | | | **Optimal Decision** |
|---|---|---|---|
| State | $j^*$ | $f_5(x_5)$ | $I_{5,j*}$ |
| $x_5$ | 2 | $1.212x_5$ | $I^*_{5,2} = x_5$ |

**Stage 4 Computations**

$$f_4(x_4) = \max_{\substack{0 \le I_{4,j} \le x_4 \\ j e \{1,2,3,4\}}} \{s_4 + f_5(x_5)\}$$

where

$$s_4 = \alpha_1^2 I_{4,1} + \alpha_2^2 I_{4,2} + \alpha_3^2 I_{4,3} + \alpha_4^2 I_{4,4} = 1.024144 I_{4,1} + 1.281424 I_{4,2} + 1.290496 I_{4,3} + 1.1449 I_{4,4}$$

$$x_5 = P_5 + q_{4,1} I_{4,1} + q_{4,2} I_{4,2} + q_{4,3} I_{4,3} + q_{4,4} I_{4,4} = \$13,000 + 0.032 \mathrm{I}_{4,1} + 0.025 I_{4,2} + 0.0175 I_{4,3} + 0.05 I_{4,4}$$

$$\Rightarrow f_4(x_4) = \max_{\substack{0 \le I_{4,j} \le x_4 \\ j \in \{1,2,3,4\}}} \{1.024144 I_{4,1} + 1.281424 I_{4,2} + 1.290496 I_{4,3} + 1.1449 I_{4,4} + 1.212(13,000$$

$$+ 0.032 I_{4,1} + 0.025 I_{4,2} + 0.0175 I_{4,3} + 0.05 I_{4,4})\}$$

$$= \max_{\substack{0 \le I_{4,j} \le x_4 \\ j \in \{1,2,3,4\}}} \{1.062928 I_{4,1} + 1.311724 I_{4,2} + 1.311706 I_{4,3} + 1.2055 I_{4,4} + 15,756\}$$

**Table 8: Summary of the Optimal Investment Policy and Return for Stage 4**

| **Optimum solution** | | | **Optimal Decision** |
|---|---|---|---|
| State | $j^*$ | $f_4(x_4)$ | $I_{4,j*}$ |
| $x_4$ | 2 | 15756+1.311724 $x_4$ | $I_{4,2} = x_4$ |

**Stage 3 Computations**

$$f_3(x_3) = \max_{0 \le I_{3,j} \le x_5} \{s_3 + f_4(x_4)\}$$

where

$$s_3 = 1.036433728 I_{3,1} + 1.450571968 I_{3,2} + 1.466003456 I_{3,3} + 1.225043 I_{3,4}$$

$$x_4 = P_4 + q_{3,1} I_{3,1} + q_{3,2} I_{3,2} + q_{3,3} I_{3,3} + q_{3,4} I_{3,4}$$

$$= 5000 + 0.03 I_{3,1} + 0.023 I_{3,2} + 0.027 I_{3,3} + 0.083 I_{3,4}$$

thus,

$$f_3(x_3) = \max_{\substack{0 \le I_{3,j} \le x_3 \\ j \in \{1,2,3,4\}}} \{1.036433728 I_{3,1} + 1.450571968 I_{3,2} + 1.466003456 I_{3,3} + 1.225043 I_{3,4}$$

$$+ 15756 + 1.311724\,(5000 + 0.03 I_{3,1} + 0.023 I_{3,2} + 0.027 I_{3,3} + 0.083 I_{3,4})\}$$

$$= \max_{\substack{0 \le I_{3,j} \le x_3 \\ j \in \{1,2,3,4\}}} \{1.0757785448 I_{3,1} + 1.48074162 I_{3,2} + 1.501420004 I_{3,3} + 1.3339916092 I_{3,4} + 22314.62\}$$

**Table 9: Summary of the Optimal Investment Policy and Return for Stage 3**

| Optimum solution | | | Optimal Decision |
|---|---|---|---|
| State | $j^*$ | $f_3(x_3)$ | $I_{3,j^*}$ |
| $x_3$ | 3 | $22314.62+1.501420004\,x_3$ | $I_{33}=x_3$ |

**Stage 2 computations**

$$f_2(x_2)=\max_{\substack{0\le I_{2,j}\le x_2\\ j\in\{1,2,3,4\}}}\{s_2+f_3(x_3)\},$$

where

$$s_2=\alpha_1^4 I_{2,1}+\alpha_2^4 I_{2,2}+\alpha_3^4 I_{2,3}+\alpha_4^4 I_{2,4}$$
$$=1.048870933I_{21}+1.642047468I_{22}+1.665379926I_{2,3}+1.31079601I_{2,4}$$

$$x_3=P_3+q_{2,1}I_{2,1}+q_{2,2}I_{2,2}+q_{2,3}I_{2,3}+q_{2,4}I_{2,4}$$
$$=12,000+0.018I_{2,1}+0.011I_{2,2}+0.03I_{2,3}+0.02I_{2,4};$$

thus,

$$f_2(x_2)=\max_{\substack{0\le I_{2,j}\le x_2\\ j\in\{1,2,3,4\}}}\{1.048870933I_{2,1}+1.642047468I_{2,2}+1.665379926I_{2,3}+1.31079601I_{2,4}+22314.62+$$
$$1.501420004\,(12,000+0.018I_{21}+0.011I_{22}+0.03I_{23}+0.02I_{24})\}$$
$$=\max_{\substack{0\le I_{2,j}\le x_2\\ j\in\{1,2,3,4\}}}\{1.0758964931I_{2,1}+1.658563088I_{2,2}+1.710422526I_{2,3}+1.34082441I_{2,4}+40331.66005\}$$

**Table 10: Summary of the Optimal Investment Policy and Return for Stage 2**

| Optimal Solution | | | Optimal Decision |
|---|---|---|---|
| State | $j^*$ | $f_2(x_2)$ | $I_{2,j^*}$ |
| $x_2$ | 3 | $40331.66005+1.710422526x_2$ | $I_{23}=x_2$ |

**Stage 1 Computations**

$$f_1(x_1)=\max_{\substack{0\le I_{1,j}\le x_1\\ j\in\{1,2,3,4\}}}\{s_1+f_2(x_2)\}$$

where,

$$s_1=\alpha_1^5 I_{1,1}+\alpha_2^5 I_{1,2}+\alpha_3^5 I_{1,3}+\alpha_4^5 I_{1,4}$$
$$=1.061457384I_{1,1}+1.858797734I_{1,2}+1.891871596I_{1,3}+1.402551731I_{1,4}$$
$$x_2=P_2+q_{1,1}I_{1,1}+q_{1,2}I_{1,2}+q_{1,3}I_{1,3}+q_{1,4}I_{1,4}$$
$$=8000+0.02I_{1,1}+0.013I_{1,2}+0.015I_{1,3}+0.022I_{1,4}$$

thus,

$$f_1(x_1) = \max_{\substack{0 \le I_{1,j} \le x_1 \\ j \in \{1,2,3,4\}}} \left\{ \begin{array}{l} 1.0614573841_{1,1} + 1.858797734 I_{1,2} + 1.891871596 I_{1,3} \\ +1.402551731 I_{1,4} + 40331.66005 \\ +1.710422526\left(8000 + 0.02 I_{1,1} + 0.013 I_{1,2} + 0.015 I_{1,3} + 0.022 I_{1,4}\right) \end{array} \right\}$$

$$= \max_{\substack{0 \le I_{1,j} \le x_1 \\ j \in \{1,2,3,4\}}} \left\{ \begin{array}{l} 1.095665835 I_{1,1} + 1.881033227 I_{1,2} + 1.917527934 I_{1,3} \\ +1.440181027 I_{1,4} + 54015.04026 \end{array} \right\}$$

**Table 11: Summary of the Optimal Investment Policy and Return for Stage 1**

| | **Optimum Solution** | **Optimal Decision** |
|---|---|---|
| State $j^*$ | $f_1(x_1)$ | $I_{1,j^*}$ |
| $x_1$ =*$*15,000*3* | 54,015.0426 +1.917527934 $x_1$ | $x_1$ =*$*15,000 =$I_{13}$ |

Working backward and noting that

$$I_{1j^*} = x_1 = \$15,000 \Rightarrow x_1 = I_{13}$$
$$I_{2j} = I_{23} = x_2, I_{3j} = I_{33} = x_{3,} I_{4j^*} = I_{42} = x_4, I_{5j^*} = I_{52} = x_5$$

$$x_2 = 8000 + 0.02(0) + 0.013(0) + 0.015(15,000) + 0.022(0) = \$8,225$$
$$x_3 = 12,000 + 0.018(0) + 0.011(0) + 0.03(8225) + 0.02(0) = \$12,246.75$$
$$x_4 = 5000 + 0.03(0) + 0.023(0) + 0.027(12,246.75) + 0.083(0) = \$5330.66225$$
$$x_5 = 13000 + 0.032 I_{4,1} + 0.025 I_{4,2} + 0.0175 I_{4,3} + 0.05 I_{4,4}$$
$$= 13000 + 0.032(0) + 0.025(5330.66225) + 0.0175(0) + 0.05(0) = \$13133.26656$$
$$s_1 = 1.061457384(0) + 1.858797734(0) + 1.891871596(15,000) + 1.402551731(0) = \$28,378.07394$$
$$s_2 = 1.048870933(0) + 1.642047468(0) + 1.665379926(8,225) + 1.31079601(0) = \$13,697.74989$$
$$s_3 = 1.036433728(0) + 1.450571968(0) + 1.466003456(12,246.75) + 1.225043(0) = \$17,953.77782$$
$$s_4 = 1.024144(0) + 1.281424(5330.66225) + 1.290496(0) + 1.1449(0) = \$6830.838543$$
$$s_5 = 1.027(0) + 1.212(13133.26656) + 1.178(0) + 0.1.7(0) = \$15,917.51907$$

TOTAL ACCUMULATION = $82,777.9526

Finally, the following problem from Taha [ ] is revisited:

### 3.2.3 Application to 2-Bank-4-Year Investment Problem

Suppose that you want to invest $ 4000 now and $2000 at the start of years 2, 3 and 4. The interest rate offered by bank 1 is 8% compounded annually and the bonuses over the 4 years are 1.8% and 1.7%, 2.1% and 2.5%, respectively. The annual interest rate offered by bank 2 is lower by 0.2% than that of bank 1, but its bonus is higher by 0.5%. The objective is to maximize the accumulated capital at the end of 4 years.

**Pertinent Remarks**

The data input operator misinterpreted/misrepresented the nominal annual interest and bonus relations between Banks 1 and 2 as follows:

$r_{i2} = r_{i1} - 0.002; q_{i2} = q_{i1} + 0.005,$ given the data set: $r_{i1} = 0.08, q_{i1} \in \{.018, .017, .021, .025\}, i \in \{1, 2, 3, 4\},$

which culminated in the following table of investment strategies and corresponding returns:

Table 12: Summary of the Optimal Investment Policy and Return for Problem 3.2.3

| Year | Optimum | Decision | Accumulatio |
|---|---|---|---|
| 1 | $I_1^* = x_1$ | Invest $x_1 = \$4000$, in First Bank | $s_1 = \$5441.80$ |
| 2 | $I_2^* = x_2$ | Invest $x_2 = \$2072$, in First Bank | $s_2 = \$2610.13$ |
| 3 | $I_3^* = 0$ | Invest $x_3 = \$2035.22$, in Secod Bank | $s_3 = \$2365.13$ |
| 4 | $I_4^* = 0$ | Invest $x_4 = \$2042.74$, in Second Bank | $s_4 = \$2274.64$ |
| **Total accumulation** $= f_1(x_1) =$ **\$12, 691.66** $(= s_1 + s_2 + s_3 + s_4)$ | | | |

**Note the following:**

$x_i = I_i + \overline{I}_i$, where $I_i$ and $\overline{I}_i$ are the amounts invested in First Bank and Second Bank respectively, at the beginning of year $i$.

The revised pertinent data for the above problem, based on the correct interpretation:

$$i \in \{1, 2, 3, 4\}; r_{i1} = 0.08; r_{i2} = 0.998 r_{i1}; q_{i1} \in \{.018, .017, .021, .025\}; q_{i2} = 1.005 q_{i1},$$

are summarized in the table below:

**Table 13: Correct Pertinent Data for Problem 3.2.3**

| | | Bank | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 1 | 2 |
| **Year** | **P** | **Nominal Annual Interest Rates** | | **End-of-Year Bonuses** | |
| **1** | 4000 | 0.08 | 0.07984 | 0.018 | 0.01809 |
| **2** | 2000 | 0.08 | 0.07984 | 0.017 | 0.01709 |
| **3** | 2000 | 0.08 | 0.07984 | 0.021 | 0.02111 |
| **4** | 2000 | 0.08 | 0.07984 | 0.025 | 0.02513 |

The stage-wise optimal investment policy and the corresponding returns are summarized in the succeeding table:

**Table 14: Summary of Optimal Investment Policy and Return for Problem 3.2.3**

| Year | Optimal Solution | Optimal Decision | Accumulation |
|---|---|---|---|
| 1 | $I_1^* = x_1$ | Invest $x_1 = \$4000,$ in Bank 1 | $s_1 = \$5,441.96$ |
| 2 | $I_2^* = 0$ | Invest $x_2 = \$2072,$ in Bank 1 | $s_2 = \$2,610.12$ |
| 3 | $I_3^* = 0$ | Invest $x_3 = \$2035.22,$ in Bank 1 | $s_3 = \$2,373.89$ |
| 4 | $I_4^* = 0$ | Invest $x_4 = \$2042.74,$ in Bank 1 | $s_4 = \$2,257.23$ |
| **Optimal Total Dollar Accumulation** $= f_1(x_1) = \$12,683.20 = \sum_{i=1}^{4} s_i$ | | | |

## 4. CONCLUSION

This article extended the results of Taha [3 ] on the optimal allocation of funds in an investment portfolio of two banks with specified nominal annual interest rates and end-of-year bonuses, over a given finite integral horizon length to an arbitrary investment portfolio of banks, with a formulation and proof of the corresponding theorem. It went further to deploy dynamic programming recursions to obtain the optimal investment policies with respect to three illustrative problems, which demonstrated the imperative for the correct interpretation of the relationships among investment parameters and the need for the automation of the computational process for the generation of the optimal investment strategies and the corresponding returns.

**References**


[1] Gupta, P.K. & Hira, D.S. (2012). Problems in operations research (Principles and Solutions). S. Chand & Company Ltd. New Delhi. 2012; 844-854.

[2] Ukwu, C. (2016). Optimal investment strategy for a certain class of probabilistic investment problems. *International Research Journal of Natural and Applied Sciences. 3(2): 92-105.*

[3] Taha, H.A. (2006). Operations Research: An Introduction. Seventh Edition. Prentice-Hall of India, New Delhi. 418-421.