International Research Journal of Natural and Applied Sciences

ISSN: (2349-4077)

Impact Factor- 5.46, Volume 5, Issue 11, November 2018

Website- www.aarf.asia, **Email** : editor@aarf.asia , editoraarf@gmail.com

FUNDAMENTAL RESULTS ON DETERMINING MATRICES FOR A CLASS OF TRIPLE-DELAY LINEAR CONTROL SYSTEMS

Ukwu Chukwunenye & Temuru, Peace Uche

Department of Mathematics, University of Jos, P.M.B. 2084,

Jos, Plateau State, Nigeria.

ABSTRACT

This research article obtainedrestricted functional forms of determining matrices for a class of triple-delay linear control systems for certain pertinent parameters, thus bridging the knowledge gap in this area of acute research need. The proofs were achieved by the exploitation of key facts about permutations, combinations of summation notations, change of variables techniques and the compositions of sigma and max functions.

Keywords: Delay, Determining, Functional, Leading, Permutation, System, Triple

1. INTRODUCTION

Controllability is a very important concept that is applied in aerospace engineering, optimal control theory, systems theory, quantum systems, power systems, industrial and chemical process controls etc. This concept was first introduced by Kalman in 1963 (Zmood, 1971) and a survey of controllability of dynamical systems was done by Klamka (2013). In addition, there have been intense research activities on qualitative approach to controllability of linear and nonlinear systems. Xianlong (2013) researched on approximate controllability of semi linear neutral retarded systems and Jackreece (2014) worked on the controllability of neutral integro-differential equations.

However, one is not aware of any other results that comprehensively interrogated the controllability of linear autonomous control systems of single-delay neutral and double-delay types via the structures of the determining matrices except (Ukwu, 2014a; 2016a). Thedetermination of the computational complexity and electronic implementations of determining matrices for double-delay linear autonomous control systems and single-delay linear autonomous neutral control systems by Ukwu (2016b) are novel contributions in this area.

In the study of Euclidean controllability of linear autonomous control systems, determining matrices are preferred veritable tools as they are the least computationally intensive, when compared to indices of control systems matrices or controllability Grammians.

Unfortunately there is no known published work that has attempted the extension of the great feats of(Ukwu, 2014a; 2016a)to delay control systems with triple time-delays in the state variables. This could be attributed to the severe difficulty in identifying recognizable mathematical patterns needed for any conjecture on functional forms of determining matrices and subsequent inductive proof. It is against this backdrop that this study makes a positive contribution to knowledge by correctly establishing relevant results on functional forms of determining matrices for the afore-mentioned triple-delay systems for certain pertinent parameters.

2. THEORETICAL UNDERPINNING

Let r_a, r_b, r_c be nonnegative integers and let $P_{a(r_a),b(r_b),c(r_c)}$ denote the set of all permutations of

$\underbrace{a,a,\ldots,a}_{r_a \text{ times}}\ \underbrace{b,b,\ldots,b}_{r_b \text{ times}}\ \underbrace{c,c,\ldots,c}_{r_c \text{ times}}$: the permutations of the objects $a,b,c,$ in which i appears r_i times, $i \in \{a,b,c\}$.

In particular, the following summarized results due toUkwu (2014b) on functional forms of determining matrices are quite relevant:

Consider the class of double-delay linear autonomous control systems of the form:

$$\dot{x}(t) = A_0\, x(t) + A_1\, x(t-h) + A_2\, x(t-2h) + Bu(t); t \geq 0 \qquad (1)$$

$$x(t) = \phi(t), t \in [-2h, 0], h > 0 \qquad (2)$$

where A_0, A_1, A_2 are $n \times n$ constant matrices with real entries, B is an $n \times m$ constant matrices with real entries.

2.1 **Determining Equations:** Uniqueness and Existence

Let $Q_k(s)$ be an $n \times n$ matrix function defined by

$$Q_k(s) = A_0 Q_{k-1}(s) + A_1 Q_{k-1}(s-h) + A_2 Q_{k-1}(s-2h) \text{ for } k = 1,2,3,\ldots s > 0,$$

with intial conditions:

$$Q_0(0) = I_n; Q_0(s) = 0; s \neq 0 .$$

These initial conditions guarantee the unique solvability of the matrix function $Q_k(s)$ (Gabasov & Kirillova, 1976; Chidume, 2007).

2.2 Computable Expression of Determining Matrices (Ukwu, 2016b)

For $0 \le j \le k \ne 0$, j, k integers,

$$Q_k(jh) = \sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1,\ldots,v_k)\in P_{0(r+k-j),\,1(j-2r),\,2(r)}} A_{v_1},\ldots,A_{v_k}.$$

For $j \ge k \ge 1$, j, k integers

$$Q_k(jh) = \begin{cases} \displaystyle\sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1,\ldots,v_k)\in P_{0(r),\,1(2k-j-2r),\,2(r+j-k)}} A_{v_1},\ldots,A_{v_k} & 1 \le j \le 2k \\ 0, \; j \ge 2k+1 \end{cases}$$

The proofs were established with the techniques of mathematical induction hypothesis, summation facts, greatest integer functions and change of variables.

Consider the class of single-delay linear neutral autonomous control systems of the form:

$$\frac{d}{dt}\left[x(t) - A_{-1}x(t-h)\right] = A_0 x(t) + A_1 x(t-h) + B\,u(t);\ t \ge 0 \quad (3)$$

$$x(t) = \phi(t),\ t \in [-h, 0],\ h > 0 \quad (4)$$

where A_{-1}, A_0, A_1 are $n \times n$ constant matrices with real entries, B is an $n \times m$ constant matrix with real entries. The initial function ϕ is in $C([-h, 0], \mathbf{R}^n)$, the space of continuous functions from $[-h, 0]$ into the real n-dimension Euclidean space, $\mathbf{R}^n$ with norm defined by $\|\phi\| = \sup_{t\in[-h,0]} |\phi(t)|$, (the sup norm). The control u is in the space $L_\infty([0, t_1], \mathbf{R}^n)$, the space of essentially bounded measurable functions taking $[0, t_1]$ into $\mathbf{R}^n$ with norm $\|\phi\| = ess\sup_{t\in[0,t_1]} |u(t)|$.

2.2.1 **Determining Equations:** Uniqueness and Existence

Let $Q_k(s)$ be an $n \times n$ matrix function defined by

$$Q_k(s) = A_{-1}Q_k(s-h) + A_0 Q_{k-1}(s) + A_1 Q_{k-1}(s-h) \text{ for } k = 1,2,3,\ldots s > 0,$$

with intial conditions:

$$Q_0(0) = I_n;\ Q_k(s) = 0, \text{ for } \min\{k,s\} < 0 .$$

These initial conditions guarantee the unique solvability of the matrix function $Q_k(s)$ (Gabasov & Kirollova, 1976; Ukwu, 2016b).

2.3 Expressions and Structures of the Determining Matrices for Single-Delay Linear Neutral Autonomous Control Systems (Ukwu, 2016b)

If j,k are positive integers, then

$$Q_k(jh) = \sum_{(v_1 \ldots v_{j-k}) \in P_{-1(j),\,0(k)}} A_{v_1} \ldots A_{v_k}$$

$$+ \left[\sum_{(v_1 \ldots v_j) \in P_{-1(j-k),1(k)}} A_{v_1} \ldots A_{vj} + \sum_{r=1}^{k-1} \sum_{(v_1,\ldots v_{j+r}) \in P_{-1(r+j-k),0(r),1(k-r)}} A_{v_1} \ldots A_{v_{j+r}} \right] \operatorname{sgn}\left(\max\{0, j+1-k\}\right)$$

$$+ \left[\sum_{(v_1,\ldots,v_k) \in P_{0(k-j),1(j)}} A_{v_1},\ldots A_{v_k} + \sum_{r=1}^{j-1} \sum_{(v,\ldots v) \in P_{-1(r),0(r+k-j),1(j-r)}} A_{v_1} \ldots A_{v_{k+r}} \right] \operatorname{sgn}\left(\max\{0, k-j\}\right)$$

3. MATERIAL AND METHODS

3.1 Identification of Work-Based Triple-Delay Linear Autonomous Control Systems

Consider the triple-delay linear autonomous control system.

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h) + A_3 x(t-3h) + Bu(t); t \ge 0 \qquad (5)$$

$x(t) = \phi(t), t \in [-3h, 0], h > 0$ (6)

where A_0, A_1, A_2 and A_3 are $n \times n$ constant matrices with real entries, B is an $n \times m$ constants matrix with real entries. The initial function ϕ is in $C([-3h, 0], \mathbf{R}^n)$, the space of continuous functions from $[-3h, 0]$ into the real n-dimensional Euclidean space, $\mathbf{R}^n$ with norm defined by $\|\phi\| = \sup_{t\in[-3h,0]}\{|\phi(t)|\}$, (the sup norm). The control u is in the space $L_\infty([0, t_1], \mathbf{R}^n)$, the space of essentially bounded measureable functions taking $[0, t_1]$ into $\mathbf{R}^n$ with norm $\|\phi\| = ess\sup_{t\in[0,t_1]}\|u(t)\|$.

Any control $u \in L_\infty([0, t_1], \mathbf{R}^n)$, will be referred to as an admissible control.

SeeChidume (2007) for further discussion on L_p (or L^p), $p \in \{1, 2, \cdots, \infty\}$.

3.1.1 **Determining Equations:** Uniqueness and Existence

Let $Q_k(s)$ be an $n \times n$ matrix function defined by

$$Q_k(s) = A_0 Q_{k-1}(s) + A_1 Q_{k-1}(s-h) + A_2 Q_{k-1}(s-2h) + A_3 Q_{k-1}(s-3h) \text{ for } k = 1, 2, 3, \ldots s > 0,$$

with intial conditions:

$$Q_0(0) = I_n; Q_0(s) = 0; s \neq 0.$$

These initial conditions guarantee the unique solvability of the matrix function $Q_k(s)$

The nextsection furnishes the functional form of the determining matrices of the system (5) on the $j-$interval $[3k-3, \infty)$.

4. RESULTS

4.1 Preliminary Result on Determining Matrices $Q_k(s), s \in \mathbf{R}$

(i) $Q_k(s) = 0$ if $s < 0$

(ii) $Q_k(0) = A_0^k$

(iii) $Q_k(s) = 0$ if $s \neq rh$ for any integer r

(iv) $Q_k(h) = \sum_{(v_1 \ldots v_k) \in P_{0(k-1),\,1(1)}} \prod_{j=1}^{k} A_{v_j}$; $k \geq 1$

(v) $Q_1(jh) = A_j \operatorname{sgn}(\max\{4-j, 0\})$

Proof

(i) Let $s < 0$.

$$k = 1 \Rightarrow Q_1(s) = A_0 Q_0(s) + A_1 Q_0(s-h) + A_2 Q_0(s-2h) + A_3 Q_0(s-3h) = A_0 0 + A_1 0 + A_2 0 + A_3 0 = 0$$

So the assertion is true for $k = 1$.

Assume that $Q_k(s) = 0$ for $s < 0$ and for $k \in \{2, \ldots, p\}$ for some integer $p \geq 2$. Then

$$Q_{p+1}(s) = A_0 Q_p(s) + A_1 Q_p(s-h) + A_2 Q_p(s-2h) + A_3 Q_p(s-3h) = A_0 0 + A_1 0 + A_2 0 + A_3 0 = 0$$

by the induction hypothesis, since $s < 0 \Rightarrow s-h < 0, s-2h < 0, s-3h < 0$. Therefore

$Q_k(s) = 0 \quad \forall s < 0$, as desired.

(ii) $Q_1(0) = A_0 Q_0(0) + A_1 Q_0(-h) + A_2 Q_0(-2h) + A_3 Q_0(-3h) = 0$

So (ii) is true for $k = 1$. Assume that

$Q_k(0) = A_0^k$ for $k \in \{2, \ldots, p\}$ for some integer $p \geq 2$. Then

$$Q_{p+1}(0) = A_0 Q_p(0) + A_1 Q_p(-h) + A_2 Q_p(-2h) + A_3 Q_p(-3h) = A_0 A_0^p = A_0^{p+1},$$

by (i) and the induction hypothesis. Therefore (ii) is proved.

(iii) Let $k = 1$ and $s \neq rh$ for any integer r; then

$Q_1(s) = A_0Q_0(s) + A_1Q_0(s-h) + A_2Q_0(s-2h) + A_3Q_0(s-3h) = 0,$ since $s \notin \{0,1,2,\cdots\}$

Assume that $Q_k(s) = 0$ for $k \in \{2,\ldots,p\}$, for some integer p. Then

$Q_{p+1}(s) = A_0Q_p(s) + A_1Q_p(s-h) + A_2Q_p(s-2h) + A_3Q_p(s-3h) = 0,$ (by the induction hypothesis)

$\Rightarrow Q_k(s) = 0 \ \forall s \neq rh,$ for any integer r.

(iv) $Q_1(h) = A_0Q_0(h) + A_1Q_0(0) + A_2Q_0(-h) + A_3Q_0(-2h) = A_1 = \sum\limits_{v_1 \in p_{0\,(1-1),1(1),2(0)}} \prod\limits_{j=1}^{1} A_{v_j}$

Similarly,

$Q_2(h) = A_0Q_1(h) + A_1Q_1(0) + A_2Q_1(-h) + A_3Q_1(-2h) = A_0A_1 + A_1A_0 + A_2 0 + A_3 0 = A_0A_1 + A_1A_0$

$$= \sum_{v_1, v_2 \in p_{0\,(1),\,1\,(1)}} \prod_{j=1}^{2} A_{v_j} \Rightarrow \text{ the lemma is true for } k = 2$$

Assume (iv) is true for $k \in \{3,\ldots,p\}$, for some integer $p \geq 3$. Then

$Q_{p+1}(h) = A_0Q_p(h) + A_1Q_p(0) + A_2Q_p(-h) + A_3Q_p(-2h)$.

$Q_p(h) = \sum\limits_{(v_1\ldots v_p) \in p_{0\,(p-1),1\,(1)}} \prod\limits_{j=1}^{p} A_{v_j}$ by the induction hypothesis

$Q_p(0) = A_0^p, Q_p(-h) = 0 = Q_p(-2h)$ by (ii) and (i) respectively.

Therefore , $Q_{p+1}(h) = A_0 \sum\limits_{(v_1\ldots v_{p+1}) \in p_{0\,(p-1),\,1\,(1)}} \prod\limits_{j=1}^{p} A_{v_j} + A_1A_0^p = \sum\limits_{(v_1\ldots v_{p+1}) \in p_{0\,(p+1),1(1)}} \prod\limits_{j=1}^{p+1} A_{v_j}$ with leading A_0

$$+ \sum_{(v_1\ldots v_{p+1}) \in p_{0\,(p),1\,(1)}} \prod_{j=1}^{p+1} A_{v_j} \text{ with leading } A_1$$

$$= \sum_{(v_1\ldots v_{p+1}) \in p_{0\,(p),1\,(1)}} \prod_{j=1}^{p+1} A_{v_j} + \sum_{(v_1\ldots v_{p+1}) \in p_{0\,(p+1-1),1\,(1)}} \prod_{j=1}^{p+1} A_{v_j}$$

So (iv) is true for $k = p+1,$ hence true for $k \in \{1,2,\ldots\}$

(v) $Q_1(0) = A_0$, $Q_1(h) = A_1$ and $Q_1(2h) = A_2$ by (ii) and (i) and the definition of Q_0.

Also, $Q_1\left(3h\right) = A_0Q_0\left(3h\right) + A_1Q_0\left(2h\right) + A_2Q_0\left(h\right) + A_3Q_0\left(0\right) = A_3$

Let $j = p \geq 4,\ p$ integer p; then

$Q_1\left(ph\right) = A_0Q_0\left[ph\right] + A_1Q_0\left(\left[p-1\right]h\right) + A_2Q_0\left(\left[p-2\right]h\right) + A_3Q_0\left(\left[p-3\right]h\right)$

$\Rightarrow Q_1\left(ph\right) = A_0 0 + A_1 0 + A_2 0 + A_3 0 = 0$ by the definition of Q_0.

4.2 Theorem on the Functional Form of $Q_k(jh)$, for $j \geq 3k-3$

$$Q_k\left(jh\right) = \begin{cases} 0 & \text{if } j \geq 3k+1 \quad \text{(i)} \\ A_3^k & \text{if } j = 3k \quad \text{(ii)} \\ \displaystyle\sum_{(v_1\ldots v_k)\in p_{2(1),\,3(k-1)}} \prod_{j=1}^{k} A_{v_j} = \sum_{r=0}^{k-1} A_3^r A_2 A_3^{k-1-r} & \text{if } j = 3k-1 \quad \text{(iii)} \\ \displaystyle\sum_{(v_1\ldots,v_k)\in p_{1(3k-1-j),3(j+1-2k)}} \prod_{j=1}^{k} A_{v_j} + \sum_{(v_1\ldots,v_k)\in p_{2(3k-j),3(j-2k)}} \prod_{j=1}^{k} A_{v_j} & \text{if } j = 3k-2 \quad \text{(iv)} \\ \displaystyle\sum_{(v_1\ldots v_k)\in p_{1(1),2(1),3(k-2)}} \prod_{j=1}^{k} A_{v_j} + \sum_{(v_1,\ldots,v_k)\in p_{0(1),\,3(k-1)}} \prod_{j=1}^{k} A_{v_j} + \sum_{(v_1,\ldots v_k)\in p_{2(3),\,3(k-3)}} \prod_{j=1}^{k} A_{v_j}, & j = 3k-3 \quad \text{(v)} \end{cases}$$

Note : For $j = 3k-3$, (v) can be also be expressed as:

$$Q_k\left(jh\right) = \sum_{r_0=0}^{1}\sum_{r_1=0}^{r_0+1} \sum_{(v_1\cdots v_k)\in p_{0(r_0),1(r_1),2(3-3r_0-2r_1),3(k-3+2r_0+r_1)}} \prod_{j=1}^{k} A_{v_j}$$

Proof

(i) $Q_k\left(\left[3k+1\right]h\right) = A_0Q_{k-1}\left(\left[3k+1\right]h\right) + A_1Q_{k-1}\left(\left[3kh\right]\right) + A_2Q_{k-1}\left(\left[3k-1\right]h\right) + A_3Q_{k-1}\left(\left[3k-2\right]h\right)$

$k = 1 \Rightarrow Q_1\left(4h\right) = A_0Q_0\left(4h\right) + A_1Q_0\left(3h\right) + A_2Q_0\left(2h\right) + A_3Q_0\left(h\right) = 0$

$= A_0 0 + A_1 0 + A_2 0 + A_3 0 = 0$

Therefore $Q_1\left(4h\right) = 0$, by (v) of theorem 4.1

$k = 2 \Rightarrow Q_2\left(7h\right) = A_0Q_1\left(7h\right) + A_1Q_1\left(6h\right) + A_2Q_1\left(5h\right) + A_3Q_1\left(4h\right) = 0$, (by (v) of the theorem 4.1).

So (i) is true for $k = 2$. Assume (i) is true for $k \in \{3,\ldots,p\}$, for some integer $p \geq 3$. Then,

$$Q_{p+1}\left(\left(3\left[p+1\right]+1\right)h\right) = A_0 Q_p\left(\left(3\left[p+1\right]+1\right)h\right) + A_1 Q_p\left(\left[3p+3\right]h\right) + A_2 Q_p\left(\left[3p+2\right]h\right) + A_3 Q_p\left(\left[3p+1\right]h\right)$$

$$= A_0 Q_p\left(\left(3\left[p+4\right]\right)h\right) + A_1 Q_p\left(\left[3p+3\right]h\right) + A_2 Q_p\left(\left[3p+2\right]h\right) + A_3 Q_p\left(\left[3p+1\right]h\right)$$

Note that $3n+4 > 3n+1 : 3n+3 > 3n+1$ and $3n+2 > 3n+1.$ Therefore, by the induction hypothesis $Q_{p+1}\left(\left(3\left[p+1\right]+1\right)h\right) = A_0 0 + A_1 0 + A_2 0 + A_3 0 = 0.$ So is (i) valid for $k = p+1,$ hence valid for all $k.$

Therefore $Q_k\left(jh\right) = 0$ for all $j \geq 3k+1$ proving (i)

(ii) Consider $Q_k\left(3kh\right),$ for $k=1;$ this yields: $Q_1\left(3h\right) = A_3$ by (v) of the theorem 4.1.

So (ii) is true for $k=1.$

Assume that (ii) is true for $k \in \{2,\ldots,p\},$ for some integer $p.$ Then,

$$Q_{p+1}\left(3\left[p+1\right]h\right) = A_0 Q_p\left(3\left[p+1\right]h\right) + A_1 Q_p\left(\left[3p+2\right]h\right) + A_2 Q_p\left(\left[3p+1\right]h\right) + A_3 Q_p\left(3ph\right)$$

Note that $Q_p\left(3ph\right) = A_3^p$ by (i) of the theorem 4.1.

Also $Q_p\left(3\left[p+1\right]h\right) = 0 = Q_p\left(\left[3p+2\right]h\right) = Q_p\left(\left[3p+1\right]h\right)$ by (i) of the theorem 4.2.

Therefore, $Q_{p+1}\left(3\left[p+1\right]h\right) = A_3 A_3^p = A_3^{p+1}.$

Hence $Q_k\left(3kh\right) = A_3^k$ for every $k \in \{1,\ldots\},$ for some integer k proving (ii).

(iii) $Q_k\left(\left[3k-1\right]h\right) = A_0 Q_{k-1}\left(\left[3k-1\right]h\right) + A_1 Q_{k-1}\left(\left[3k-2\right]h\right) + A_2 Q_{k-1}\left(\left[3k-3\right]h\right) + A_3 Q_{k-1}\left(\left[3k-4\right]h\right)$

$k=1 \Rightarrow Q_1\left(2h\right) = A_0 Q_0\left(2h\right) + A_1 Q_0\left(h\right) + A_2 Q_0\left(0\right) + A_3 Q_0\left(-h\right) = A_2$

Assume that (iii) is true for $k \in \{2,\ldots,p\}$, for some integer $p.$

Then, $Q_{p+1}\left(\left[3\left(p+1\right)-1\right]h\right) = Q_{p+1}\left(\left[3p+2\right]h\right)$

$$= A_0 Q_p\left(\left[3p+2\right]h\right) + A_1 Q_p\left(\left[3p+1\right]h\right) + A_2 Q_p\left(\left[3ph\right]\right) + A_3 Q_p\left(\left[3p-1\right]h\right)$$

Note that $Q_p\left(\left[3p+2\right]h\right) = 0 = Q_p\left(\left[3p+1\right]\right)$ by (i) and $Q_p\left(3ph\right) = A_3^p$ by (ii).

By the induction hypothesis, $Q_p\left(\left[3p-1\right]h\right) = \sum_{\left(v_1,\ldots,v_p\right) \in P_{2(1),1(p-1)}} \prod_{j=1}^{k} A_{v_j}$

But $A_2 = Q_1\left(2h\right)$ by (v) of the theorem 4.1.

Now,

$$\sum_{(v_1,\dots v_k)\in p_{2(1),1(k-1)}} \prod_{j=1}^{k} A_{v_j} = \sum_{v_2\in p_{2(1)}} A_2 = A_2 \quad \text{for } k=1 \text{. So (iii) is true for } k=1.$$

Therefore, $Q_{p+1}\left(\left[3\left(p+1\right)-1\right]h\right) = A_2A_3^p + A_3 \sum_{(v_1,\dots,v_p)\in p_{2(1),1(p-1)}} \prod_{j=1}^{p} A_{v_j} = A_2A_3^p + \sum_{(v_1,\dots v_p)\in p_{2(1),1(p+1-1)}} \prod_{j=1}^{p+1} A_{v_j}$ with leading A_3 in each permutation of $A_{v_j\text{'s}}$, $j\in\{1,\dots,p+1\}$. Since A_2 appears only once in each permutation it can only lead once in $A_2A_3^p$ and will take the positions $2,\dots,p+1$ in other permutations. Therefore, $Q_{p+1}\left(\left[3\left(p+1\right)-1\right]h\right) = \sum_{(v_1,\dots,v_{p+1})\in p_{2(1),1(p+1)}} \prod_{j=1}^{p+1} A_{v_j}$, proving the left-hand side of (iii), that is, $Q_k\left(\left[3k-1\right]h\right) = \sum_{(v_1,\dots,v_k)\in p_{2(1),1(k-1)}} \prod_{j=1}^{k} A_{v_j}$. To prove the right-hand side of (iii), notice that $\sum_{r=0}^{k-1} A_3^r A_2 A_3^{k-1-r}$ is the sum of the permutations of A_2 and A_3.

When $r=0$, the permutation equals $\sum_{r=0}^{k-1} A_2A_3^{k-1}$, indicating A_2 leads; similarly, when $r=0$ the permutation equals $\sum_{r=1}^{k-1} A_3A_2A_3^{k-2}$ indicating A_2 takes the second position; continuning in this fashion, we deduce that for $r = k-1$, the right hand side of (iii) equals $\sum_{r=k-1}^{k-1} A_3^{k-1}A_2$, implying that A_2 trails. Note that A_2 occurs once in each permutation, while A_3 occurs $k-1$ times. This completes the proof of (iii).

(iv) $$\sum_{(v_1\dots v_k)\in p_{1(1),3(k-1)}} \prod_{j=1}^{k} A_{v_j} + \sum_{(v_1\dots v_k)\in p_{2(2),3(k-2)}} \prod_{j=1}^{k} A_{v_j}$$

Let us consider the expression $Q_k\left(\left[3k-2\right]h\right)$, for $k=1$; this yields $Q_1\left(h\right)=A_1$ by (v) of theorem 4.1. Following the mathematical convention of setting infeasible sums to zero, we deduce that for $k=1$ and $j=3k-2$ (iv) yields

$$\sum_{v_1\in p_{1(1),3(0)}} \prod_{j=1}^{1} A_{v_j} + \sum_{v_2,v_3\in p_{2(2),3(-1)}} \prod_{j=1}^{1} A_{v_j} = \sum_{v_1\in p_{1(1),3(0)}} \prod_{j=1}^{1} A_{v_j} + 0 = A_1$$

Similarly, $k=2 \Rightarrow j=3(2)-2=4$

$\Rightarrow Q_k\left([3k-1]h\right)=Q_2\left(4h\right)=A_1A_3+A_2^2+A_3A_1$, the right hand side of (iv) yields

$$\sum_{(v_1,v_3)\in p_{1(1),3(2-1)}}\prod_{j=1}^{2}A_{v_j}+\sum_{(v_2,v_3)\in p_{2(2),3(2-2)}}\prod_{j=1}^{2}A_{v_j}=\sum_{(v_1,v_3)\in p_{1(1),3(1)}}\prod_{j=1}^{2}A_{v_j}+\sum_{v_2\in p_{2(2),}}\prod_{j=1}^{2}A_{v_j}$$

$$=A_1A_3+A_2^2+A_3A_1$$

So (iv) is true for $k=2$. The rest of the proof is by mathematical induction on k .

Assume the validity of (iv) for $k\in\{3,\ldots,p\}$, for some integer $p\geq 3$. Then

$Q_{p+1}\left([3(p+1)-2]h\right)=Q_{p+1}\left([3p+1]h\right)$. But

$Q_{p+1}\left([3p+1]h\right)=A_0Q_p\left([3p+1]h\right)+A_1Q_p\left(3ph\right)+A_2Q_p\left([3p-1]h\right)+A_3Q_p\left([3p-2]h\right)$

Now $Q_p\left([3p+1]h\right)=0$ by (i); $Q_p\left(3ph\right)=A_3^p$, by (ii) ;

$Q_p\left([3p-1]h\right)=\sum_{(v_1,\ldots v_k)\in p_{2(1),3(k-1)}}\prod_{j=1}^{k}A_{v_j}=\sum_{r=0}^{k-1}A_3^rA_2A_3^{k-1-r}$ by (iii)

$Q_p\left([3p-2]h\right)=\sum_{(v_1,\ldots v_p)\in p_{1(1),3(p-1)}}\prod_{j=1}^{p}A_{v_j}+\sum_{(v_1,\ldots v_p)\in p_{2(2),3(p-2)}}\prod_{j=1}^{p}A_{v_j}$ (by the induction hypothesis).

Therefore

$$Q_{p+1}\left([3p+1]h\right)$$

$$=A_0 0+A_1A_3^p+A_2\sum_{(v_1,\ldots,v_p)\in p_{2(1),3(p-1)}}\prod_{j=1}^{p}A_{v_j}+A_3\sum_{(v_1,\ldots,v_p)\in p_{1(1),3(p-1)}}\prod_{j=1}^{p}A_{v_j}+A_3\sum_{(v_1,\ldots,v_p)\in p_{2(2),3(p-2)}}\prod_{j=1}^{p}A_{v_j}$$

$A_1A_3^p=\sum_{(v_1,\ldots v_{p+1})\in p_{1(1),3(p)}}\prod_{j=1}^{p+1}A_{v_j}$ (with leading A_1);

$A_3^pA_1=\sum_{(v_1,\ldots v_{p+1})\in p_{1(1),3(p)}}\prod_{j=1}^{p+1}A_{v_j}$ (with leading A_3)

$A_3 \sum_{(v_1,\dots,v_p)\in p_{1(1),3(p-1)}} \prod_{j=1}^{p} A_{v_j} = \sum_{(v_1,\dots,v_{p+1})\in p_{1(1),3(p)}} \prod_{j=1}^{p+1} A_{v_j}$, (with leading A_3) ,

$A_2 \sum_{(v_1,\dots,v_p)\in p_{2(1),3(p-1)}} \prod_{j=1}^{p} A_{v_j} = \sum_{(v_1,\dots,v_{p+1})\in p_{2(2),3(p-1)}} \prod_{j=1}^{p+1} A_{v_j}$.

Therefore, $A_1A_3^p + A_3 \sum_{(v_1,\dots,v_p)\in p_{21(1),3(p-1)}} \prod_{j=1}^{p} A_{v_j} + A_3 = \sum_{(v_1,\dots,v_{p+1})\in p_{1(1),3(p)}} \prod_{j=1}^{p+1} A_{v_j}$,

implying that $Q_{p+1}\left(\left[3(p+1)-2\right]h\right) = \sum_{(v_1,\dots v_{p+1})\in p_{1(1),3(p)}} \prod_{j=1}^{p+1} A_{v_j} + \sum_{(v_1,\dots v_{p+1})\in p_{2(2),(3[p+1]-2)}} \prod_{j=1}^{p+1} A_{v_j}$

(v) For $k=1$ and j=3k-3 the left hand side of the expression (v) gives $Q_1(0) = A_0$;

the right hand side equals $\sum_{v_1\in p_{1(1),2(1),3(1-2)}} \prod_{j=1}^{1} A_{v_j} + \sum_{v_1\in p_{0(1),,3(1-1)}} \prod_{j=1}^{1} A_{v_j} + \sum_{v_1\in p_{2(3),3(1-3)}} \prod_{j=1}^{1} A_{v_j}$.

Notice that $\sum_{v_1\in p_{1(1),2(1),3(-1)}} \prod_{j=1}^{1} A_{v_j}$ *and* $\sum_{v_1\in p_{2(3),3(-2)}} \prod_{j=1}^{1} A_{v_j}$ are infeasible sums, and are therefore equal to 0. Hence, the right hand side of (v) yields $\sum_{v_1\in p_{0(1)}} \prod_{j=1}^{1} A_{v_j} = A_0$. So (v) is true for $k=1$.

Similarly, for $k=2$ the right hand side $Q_k\left(\left[3k-3\right]h\right)$ yields $Q_2(3h) = A_0A_3 + A_1A_2 + A_2A_1 + A_3A_0$ (from the determining equation for $Q_k(s)$).

The right hand side of (v) with $k=2 \Rightarrow j = 3(2)-3 = 3$ gives

$$\sum_{(v_1,v_2)\in p_{1(1),2(1),3(2-2)}} \prod_{j=1}^{2} A_{v_j} + \sum_{(v_0,v_3)\in p_{0(1),,3(2-1)}} \prod_{j=1}^{2} A_{v_j} + \sum_{(v_2,v_3)\in p_{2(3),3(2-3)}} \prod_{j=1}^{2} A_{v_j}$$

the last sum $\sum_{(v_2,v_3)\in p_{2(3),3(2-3)}} \prod_{j=1}^{2} A_{v_j}$ is an infeasible sum

Accordingly the right hand side of $Q_2(3h)$ gives

$$\sum_{(v_1,v_2)\in P_{1(1),2(1)}} \prod_{j=1}^{2} A_{v_j} + \sum_{(v_0,v_3)\in P_{0(1),,3(1)}} \prod_{j=1}^{2} A_{v_j} = A_0A_3 + A_1A_2 + A_2A_1 + A_3A_0,$$

which is consistent with the expression for $Q_2(3h)$. So (v) is valid for $k=2$.

Assume the validity of (v) for $k \in \{3,\ldots,p\}$, for some integer $p \geq 3$.

Then, $Q_{p+1}([3(p+1)-3]h) = Q_{p+1}(3ph)$. And $Q_{p+1}(3ph)$

$= A_0Q_p(3ph) + A_1Q_p([3p-1]h) + A_2Q_p([3p-2]h) + A_3Q_p([3p-3]h)$.

$Q_{p+1}(3ph) = A_3^p$, by (ii) of the theorem; $Q_p([3p-1]h) = \sum_{(v_1,..v_p)\in P_{2(1),3(p-1)}} \prod_{j=1}^{p} A_j = \sum_{r=0}^{p-r} A_3^r A_2 A_3^{p-1-r}$ by (iii) of the theorem.

$$Q_p([3p-2]h) = \sum_{(v_1,..v_p)\in P_{1(1),3(p-1)}} \prod_{j=1}^{p} A_j + \sum_{(v_1,..v_p)\in P_{2(2),3(p-2)}} \prod_{j=1}^{p} A_j \text{ by (iv) of the theorem}$$

$$Q_p([3p-3]h) = \sum_{(v_1,..v_p)\in P_{1(1),2(1),3(p-2)}} \prod_{j=1}^{p} A_j + \sum_{(v_1,..v_p)\in P_{0(1),3(p-1)}} \prod_{j=1}^{p} A_j + \sum_{(v_1,..v_p)\in P_{2(3),3(p-3)}} \prod_{j=1}^{p} A_j$$

(by the induction hypothesis).

Therefore,

$$\begin{aligned} Q_{p+1}(3ph) = {} & A_0A_3^p + A_1 \sum_{(v_1,..v_p)\in P_{2(1),3(p-1)}} \prod_{j=1}^{p} A_j + A_2 \sum_{(v_1,..v_p)\in P_{1(1),3(p-1)}} \prod_{j=1}^{p} A_j \\ & + A_2 \sum_{(v_1,..v_p)\in P_{2(2),3(p-2)}} \prod_{j=1}^{p} A_j + A_3 \sum_{(v_1,..v_p)\in P_{1(1),2(1),3(p-2)}} \prod_{j=1}^{p} A_j \\ & + A_3 \sum_{(v_1,..v_p)\in P_{0(1),3(p-1)}} \prod_{j=1}^{p} A_j + A_3 \sum_{(v_1,..v_p)\in P_{2(3),3(p-3)}} \prod_{j=1}^{p} A_j \end{aligned}$$

Now, $A_0 A_3^p = \sum_{(v_1,\ldots,v_{p+1}) \in p_{0(1),3(p)}} \prod_{j=1}^{p+1} A_{v_j}$ (with trailing A_3), similarly

$$A_3 \sum_{(v_1,\ldots,v_p) \in p_{0(1),3(p-1)}} \prod_{j=1}^{p} A_{v_j} = \sum_{(v_1,\ldots,v_{p+1}) \in p_{0(1),3(p)}} \prod_{j=1}^{p+1} A_{v_j} \quad \text{(with leading } A_3\text{)}.$$

Therefore, $A_0 A_3^p + A_3 \sum_{(v_1,\ldots,v_p) \in p_{0(1),3(p-1)}} \prod_{j=1}^{p} A_{v_j} = \sum_{(v_1,\ldots,v_{p+1}) \in p_{0(1),3(p)}} \prod_{j=1}^{p+1} A_{v_j}$.

Also $A_2 \sum_{(v_1,\ldots,v_p) \in p_{2(2),3(p-2)}} \prod_{j=1}^{p} A_{v_j} = \sum_{(v_1,\ldots,v_{p+1}) \in p_{2(3),3(p-2)}} \prod_{j=1}^{p+1} A_{v_j}$ (with leading A_2),

$$A_3 \sum_{(v_1,\ldots,v_p) \in p_{2(3),3(p-3)}} \prod_{j=1}^{p} A_{v_j} = \sum_{(v_1,\ldots,v_{p+1}) \in p_{2(3),3(p-2)}} \prod_{j=1}^{p+1} A_{v_j},$$

$$A_1 \sum_{(v_1,\ldots,v_p) \in p_{2(1),3(p-1)}} \prod_{j=1}^{p} A_{v_j} = \sum_{(v_1,\ldots,v_{p+1}) \in p_{2(1),3(p-1)}} \prod_{j=1}^{p+1} A_{v_j} \quad \text{(with leading } A_1\text{)}.$$

Consequently,

$$Q_{p+1}(3ph) = \sum_{(v_1,\ldots v_{p+1}) \in p_{0(1),3(p)}} \prod_{j=1}^{p+1} A_j + \sum_{(v_1,\ldots v_{p+1}) \in p_{2(1),3(p-1)}} \prod_{j=1}^{p+1} A_j + \sum_{(v_1,\ldots v_{p+1}) \in p_{2(2),3(p-2)}} \prod_{j=1}^{p+1} A_j$$

$$+ \sum_{(v_1,\ldots v_{p+1}) \in p_{2(2),3(p-2)}} \prod_{j=1}^{p+1} A_j + \sum_{(v_1,\ldots v_{p+1}) \in p_{1(1),3(p-1)}} \prod_{j=1}^{p+1} A_j + \sum_{(v_1,\ldots v_{p+1}) \in p_{1(1),2(2),3(p-1)}} \prod_{j=1}^{p+1} A_j$$

Therefore,

$$Q_{p+1}(3ph) = \sum_{(v_1,\ldots v_{p+1}) \in p_{0(1),3(p)}} \prod_{j=1}^{p+1} A_j + \sum_{(v_1,\ldots v_{p+1}) \in p_{2(2),3(p-2)}} \prod_{j=1}^{p+1} A_j, \text{(with leading } A_2\text{)}$$

$$+\sum_{(v_1,..v_{p+1})\in p_{2(2),3(p-2)}} \prod_{j=1}^{p+1} A_j, \qquad \text{(with leading } A_3\text{)}$$

$$+\sum_{(v_1,..v_{p+1})\in p_{2(1),3(p-1)}} \prod_{j=1}^{p+1} A_j \qquad \text{(with leading } A_1\text{)}$$

$$+\sum_{(v_1,..v_{p+1})\in p_{1(1),3(p-1)}} \prod_{j=1}^{p+1} A_j \qquad \text{(with leading } A_2\text{)}$$

$$+\sum_{(v_1,..v_{p+1})\in p_{1(1),2(2),3(p-1)}} \prod_{j=1}^{p+1} A_j \qquad \text{(with leading } A_3\text{)} \qquad (4.1)$$

5. SUMMARY

This research article has successfully carried out two tasks. First, it investigated and established functional expressions and structure of determining matrices $Q_k(s), s \in \mathbf{R}$, of triple-delay systems for $s = jh$, $\min\{j,k\} = 1, j,k$ integers and for $s \in \{rh\} \cup \mathbf{R}$, $r \notin \{\cdots,-2,-1,1,2,\cdots\}$. The article proceeded to obtain thefunctional expressions and structure of the determining matrices $Q_k(jh)$ on the j-interval $[3k-3,\infty)$. The formulation and proof of the expressions for the determining matrices of the system of interest were achieved by the exploitation of key facts about permutations, the interrogation of the feasibility dispositions of $Q_k(jh)$ and the application of the principle of mathematical induction. The work has added to the body of knowledge and has provided much needed impetus for further work on determining matrices of system (5). A sequel to this article will investigate more general results.

REFERENCES

Chidume, C. (2007). Applicable Functional Analysis, *the Abdus Salam, International Centre for Theoretical Physics, Trieste, Italy*.

Gabasov, R. &Kirillova, F. (1976).*The qualitative theory of optimal processes.*Marcel Dekker Inc., New York.

Jackreece, P.C. (2014). Controllability of Neutral Integro-differential equation with Infinite Delay. *International Journal of Mathematics and Statistics Inventions (IJMSI), 2(4), 58-66*.

Klamka, J. (2013). Controllability of Dynamical Systems: A survey. *Bulletin of the Polish Academy of Sciences: Technical Science.doi:10.2478/bpasts-2013-0031*

Ukwu, C. (2014a). The Structure of Determining Matrices for a Class of Double-Delay Control Systems.*International Journal of Mathematics and Statistics Inventions (IJMSI), 2(2),14-30*.

Ukwu, C. (2014b). Necessary and Sufficient Conditions for Controllability of Single-Delay Autonomous Neutral Control Systems and Application. *International Journal of Mathematics and Statistics Inventions (IJMSI), 2 (5), 30-35*.

Ukwu, C. (2016a). Necessary and Sufficient Conditions for Controllability of Double-Delay Autonomous Linear Control Systems. *Journal of Scientific Research and Reports, 10(3), 1-9*. doi:10.9734/JSRR/2016/24348.

Ukwu, C. (2016b). *Functional Differential Systems: Structures, Cores and Controllability*. Lap Lambert Academic Publishing.

Xianlong, F. (2013). Approximate Controllability of Neutral Semi- linear Retarded Systems. *IMA Journal of Mathematical Control and Information, doi: 10, 1093/imamci/dnt019*

Zmood, R. B. (1971). *On Euclidean Space and Function Space Controllability of Control Systems with Delay*.Technical Report, the University of Michigan, USA.