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# FIXED POINT RESULTS IN NORMED LINEAR SPACE

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## ABSTRACT

*In this paper, we prove a common fixed point theorem for six mappings which satisfying compatible of type (A) under an implicit relation and rational expression.*

**MATHEMATICAL SUBJECT CLASSIFICATION (2000) -** 54H25, 47H10

**KEY WORDS AND PHRASES:** Normed linear space, Compatible mappings of type (A), Common fixed point.

## 1 INTRODUCTION AND PRELIMINARIES

Several authors proved common fixed point theorems using the concept of compatible maps as compatible of type (A) [1] and compatible of type (B) [3]. In 1998, H.K. Pathak, Y.J.

Cho, S.M. Kang , B. Madharia [2] introduced another extension of compatible mapping of type (A) in normed spaces , called compatible mappings of type (C) and with some examples they compared these mappings with compatible maps , compatible maps of type (A) and compatible maps of type (B) . Further Popa [4], did lot of work for compatible mappings satisfying an implicit relation. In the continuation of this context we are proving a common fixed point theorem with six mappings which satisfying the compatible mappings of type (A) and implicit function in $(R^{+})^{8}$ .

**IMPLICIT RELATION**

As in [4], we denote by $F$ the set of all real continuous functions $F: (R+)^{8} \rightarrow R$

(F1): $F$ is non – increasing in the variable $t_4$

(F2): there exists $h \in (0,1)$ such that for every $u, v \geq 0$ with

(F3): $F(u, 0, u, u, u, 0,0,0) > 0$

(F4): $F(u, 0,0,0, u, 0, u, 0) > 0$

(F5): $F(0, u, 0, u, 0,0,0, u) > 0$

(F6): $F(0, u, 0, u, u, u, 0,0) > 0$

(F*): $F(0, u, v, u + v, 0,0, v, u) \leq 0.$

Then we have $u \leq hv$

**Definition1.1**. Let $S \ and \ T$ be mappings from a metric space $(X, d)$ into itself. The mapping $S \ and \ T$ are said to be compatible, if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

**Definition1.2**. The mappings $S \, and \, T$ from metric space $(X, d)$ into itself are said to be compatible of type (A) if

$$\lim_{n\to\infty} d(STx_n, TTx_n) = 0 \text{ and } \lim_{n\to\infty} d(TSx_n, SSx_n) = 0$$

**Proposition 1.** Let $S\ and\ T$ be continuous mappings from a metric space itself. Then the following are equivalent:

(i) $S\ and\ T$ are compatible

(ii) $S\ and\ T$ are compatible of type (A)

**Proposition 2.** Let $S\ and\ T$ be mappings from a metric space $(X, d)$ into itself. If $S\ and\ T$ are compatible of type (A) in $X$ such that $\lim_{n\to\infty} S\,x_n = \lim_{n\to\infty} T\,x_n = z$ for some $z \in X$, then

(i) $\lim_{n\to\infty} TS\,x_n = Tz$ if T is continuous at $z$,

(ii) $STz = TSz$ and $Sz = Tz$, if $S\ and\ T$ are continuous at $z$,

## 2. Main results

**Theorem 2.1:** Let $A, B, S, T, I\ \&\ J$ be mappings from a Normed linear space $(X, \|\ \|)$ into itself satisfying the conditions

(a) $I(X) \subset AB(X),\ J(X) \subset ST(X)$

(b) One of $A, B, S, T, I, J$ is continuous

(c) The pair $(I, AB), (J, ST)$ are compatible of type (A)

(d) $F\left\{\begin{matrix} \|STy - Ix\|, \|STy - Jy\|, \|Bx - STy\|, \|Bx - Jy\|, \|By - Ix\|, \|STy - ABy\|, \\ \|ABx - Ix\|, \|ABy - Jy\| \end{matrix}\right\} \leq 0$

for all $x, y\ in\ X$,

$then\ A, B, S, T, I\ \&\ J$ have a unique common fixed point.

**Proof:** By (a) $I(X) \subset AB(X)$, for any $x_0 \in X$ there exist a point $x_1 \in X$ such that $Ix_0 = ABx_1$. Since $J(X) \subset ST(X)$, for this point $x_1$ we choose a point $x_2 \in X$ such that $Ix_1 = STx_2$ .Inductively we can find a sequence

$$y_{2n} = Ix_{2n} = ABx_{2n+1}$$

$$y_{2n+1} = Jx_{2n+1} = \mathrm{ST}x_{2n+2}$$

Using inequality (d), we have successively

$$F\begin{Bmatrix} \|STx_{2n+2}-Ix_{2n+1}\|, \|STx_{2n+2}-Jx_{2n+2}\|, \|ABx_{2n+1}-STx_{2n+2}\|, \\ \|ABx_{2n+1}-Jx_{2n+2}\|, \|ABx_{2n+2}-Ix_{2n+1}\|, \|STx_{2n+2}-ABx_{2n+2}\|, \\ \|ABx_{2n+1}-Ix_{2n+1}\|, \|ABx_{2n+2}-Jx_{2n+2}\| \end{Bmatrix} \leq 0$$

$$= F\begin{Bmatrix} \|y_{2n+1}-y_{2n+1}\|, \|y_{2n+1}-y_{2n+2}\|, \|y_{2n}-y_{2n+1}\|, \|y_{2n}-y_{2n+2}\|, \\ \|y_{2n+1}-y_{2n+1}\|, \|y_{2n+1}-y_{2n+1}\|, \|y_{2n}-y_{2n+1}\|, \|y_{2n+1}-y_{2n+2}\| \end{Bmatrix} \leq 0$$

By condition ($F_1$), we have

$$F\begin{Bmatrix} 0, \|y_{2n+1}-y_{2n+2}\|, \|y_{2n}-y_{2n+1}\|, \|y_{2n}-y_{2n+1}\| + \|y_{2n+1}-y_{2n+2}\|, 0,0, \\ \|y_{2n}-y_{2n+1}\|, \|y_{2n+1}-y_{2n+2}\| \end{Bmatrix} \leq 0$$

So we obtain by ($F^*$),

$$\|y_{2n+1}-y_{2n+2}\| \leq \mathrm{h}\|y_{2n}-y_{2n+1}\|$$

Similarly, we get

$$\|y_{2n}-y_{2n+1}\| \leq \mathrm{h}\|y_{2n+1}-y_{2n}\|$$

Proceeding in the same way, we get

$$\|y_{2n+1}-y_{2n+2}\| \leq \mathrm{h}^{2\mathrm{n}-1}\|y_0-y_1\|$$

It follows that $\{y_n\}$ is a cauchy sequence in $X$ is complete, $\{y_n\}$ is convergent to a point $z\ in\ X$. Since $Ix_{2n}, Jx_{2n+1}, ABx_{2n+1,}STx_{2n+2}$ are subsequences of $\{y_n\}$, they also converge to a point $z$, that is as $\rightarrow \infty$, $Ix_{2n},\ Jx_{2n+1},\ STx_{2n+1} \rightarrow z$.

Suppose $AB$ is continuous and the pair $\{I, AB\}$ is *compatible* of type (A), by proposition (2)

$I(AB)x_{2n+1} \rightarrow ABz$, $(AB^2)x_{2n+1} \rightarrow ABz$

Put $x = ABx_{2n+1}$ and $y = x_{2n+2}$ in (d)

$$F\begin{Bmatrix} \|STx_{2n+2}-IABx_{2n+1}\|, \|STx_{2n+2}-Jx_{2n+2}\|, \|(AB)^2x_{2n+1}-STx_{2n+2}\|, \\ \|(AB)^2x_{2n+1}-Jx_{2n+2}\|, \|ABx_{2n+2}-IABx_{2n+1}\|, \|STx_{2n+2}-ABx_{2n+2}\|, \\ \|(AB)^2x_{2n+1}-IABx_{2n+1}\|, \|ABx_{2n+2}-Jx_{2n+2}\| \end{Bmatrix} \leq 0$$

Which implies that, as $n \rightarrow \infty$

$$F\{\|z-ABz\|, 0, \|ABz-z\|, \|ABz-z\|, \|z-ABz\|, 0,0,0\} \leq 0$$

Which is contradiction of ($F_3$), if $\|z-ABz\| \neq 0$. Thus $= z$ .

Put $x = Ix_{2n}$ and $y = x_{2n+1}$ in (d)

$$F\begin{Bmatrix} \|STx_{2n+1} - I(Ix_{2n})\|, \|STx_{2n+1} - Jx_{2n+1}\|, \|AB(Ix_{2n}) - STx_{2n+1}\|, \\ \|AB(Ix_{2n}) - Jx_{2n+1}\|, \|ABx_{2n+1} - I(Ix_{2n})\|, \|ABx_{2n+1} - Jx_{2n+1}\|, \\ \|STx_{2n+1} - ABx_{2n+1}\|, \|AB(Ix_{2n}) - (Ix_{2n})\| \end{Bmatrix} \leq 0$$

Which implies that, as $n \to \infty$

$$F\{\|z - Iz\|, 0, \|ABz - z\|, \|ABz - z\|, \|z - Iz\|, 0, 0, \|ABz - Iz\|\} \leq 0$$

Which implies that $Iz = z = ABz$

Now we show that $Bz = z$ . By putting $x = Bz \ and \ y = x_{2n+1}$ in (d)

$$F\begin{Bmatrix} \|STx_{2n+1} - IBz\|, \|STx_{2n+1} - Jx_{2n+1}\|, \|AB(Bz) - STx_{2n+1}\|, \\ \|AB(Bx) - Jx_{2n+1}\|, \|ABx_{2n+1} - IBz\|, \|STx_{2n+1} - ABx_{2n+1}\|, \\ \|AB(Bx) - IBz\|, \|ABx_{2n+1} - Jx_{2n+1})\| \end{Bmatrix} \leq 0$$

Which implies that, as $n \to \infty$

$$F\{\|z - Bz\|, 0, 0, 0, \|z - Bz\|, 0, \|z - Bz\|, 0\} \leq 0$$

Which implies that $Bz = z, \ since \ ABz = z, Az = z$

Now the pair $\{J, ST\}$ is compatible of type (A) therefore by proposition 2

Now by putting $x = z \ and \ y = STx_{2n+2}$ in (d)

$$F\begin{Bmatrix} \|(ST)^2x_{2n+2} - Iz\|, \|(ST)^2x_{2n+2} - J(ST)x_{2n+2}\|, \|ABz - (ST)^2x_{2n+2}\|, \\ \|ABz - J(ST)x_{2n+2}\|, \|AB(ST)x_{2n+2} - Iz\|, \|(ST)^2x_{2n+2} - AB(ST)x_{2n+2}\|, \\ \|ABz - Iz\|, \|AB(ST)x_{2n+2} - J(ST)x_{2n+2}\| \end{Bmatrix} \leq 0$$

Which implies that, as $n \to \infty$

$$F\{0, \|z - STz\|, 0, \|z - STz\|, 0, 0, 0, \|z - STz\|\} \leq 0$$

Which implies that $STz = z$

Now by putting $x = z \ and \ y = Jx_{2n+1}$ in (d)

$$F\begin{Bmatrix} \|ST(Jx_{2n+1}) - Iz\|, \|ST(Jx_{2n+1}) - J(Jx_{2n+1})\|, \|ABz - ST(Jx_{2n+1})\|, \\ \|ABz - J(Jx_{2n+1})\|, \|AB(Jx_{2n+1}) - Iz\|, \|ST(Jx_{2n+1}) - AB(Jx_{2n+1})\|, \|ABz - Iz\|, \\ \|AB(Jx_{2n+1}) - J(Jx_{2n+1})\| \end{Bmatrix} \leq 0$$

Which implies that, as $n \to \infty$

$$F\{0, \|z - Jz\|, 0, \|z - Jz\|, 0, 0, 0, \|z - Jz\|\} \leq 0$$

Which implies that $Jz = z$ and hence $STz = z = Jz$

Finally we show that $Tz = z$ .Put $x = z$ and $y = Tz$ in (d)

$$F\left\{\begin{matrix} \|ST(\mathrm{Tz}) - I\mathrm{z}\|, \|ST(Tz) - J(Tz)\|, \|AB\mathrm{z} - ST(Tz)\|, \|AB\mathrm{z} - J(Tz)\|, \\ \|AB(Tz) - I\mathrm{z}\|, \|ST(Tz) - AB(Tz)\|, \|AB\mathrm{z} - I\mathrm{z}\|, \|AB(Tz) - J(Tz)\| \end{matrix}\right\} \leq 0$$

as $n \to \infty$

$F\{0, \|\mathrm{z} - Tz\|, 0, \|\mathrm{z} - Tz\|, \|Tz - \mathrm{z}\|, \|\mathrm{z} - Tz\|, 0,0\} \leq 0$

Which implies that $Tz = z$ since $STz = z, we\ have\ Sz = z$

Therefore by combining the above results, we have

$Az = Bz = Sz = Tz = Iz = Jz = z$

That is $z$ is common fixed point of $A, B, S, T, I\ and\ J$.

$$F\left\{\begin{matrix} \|STw - Iz\|, \|STw - J\mathrm{z}\|, \|AB\mathrm{z} - STw\|, \|AB\mathrm{z} - Jw\|, \|ABw - Iz\|, \\ \|STw - ABw\|, \|AB\mathrm{z} - Iz\|, \|ABw - Jw\| \end{matrix}\right\} \leq 0$$

$F\{\|w - z\|, \|w - z\|, \|z - w\|, \|z - w\|, \|w - z\|, 0,0,0\} \leq 0$

Therefore we have $w = z$ This complete the proof.

**Theorem 2.2:** Let $A, B, S, T, I$ & $J$ be mappings from a Hilbert space $(X, \|\ \|)$ into itself satisfying the conditions

(a) $I(X) \subset AB(X)$ , $J(X) \subset ST(X)$
(b) One of $A, B, S, T, I, J$ is continuous
(c) The pair $(I, AB), (J, ST)$ are compatible of type (A)
(d) $F\left\{\begin{matrix} \|ST\mathrm{y} - Ix\|^2, \|ST\mathrm{y} - J\mathrm{y}\|^2, \|Bx - ST\mathrm{y}\|^2, \|Bx - J\mathrm{y}\|^2, \|B\mathrm{y} - Ix\|^2 \\ \|ST\mathrm{y} - \mathrm{ABy}\|^2 \|ABx - Ix\|^2, \|AB\mathrm{y} - J\mathrm{y}\|^2 \end{matrix}\right\} \leq$
$0\ \ for\ all\ x, y\ in\ X,$
$then\ A, B, S, T, I$ & $J$ have a unique common fixed point.

**Proof:** Same as Theorem 2.1 with parallelogram law.

**Corollary 2.2.1:** Let $A, B, S, T, I$ & $J$ be mappings from a Hilbert space $(X, \|\ \|)$ into itself satisfying the conditions

(a) $I(X) \subset A(X)$ , $J(X) \subset S(X)$

(b) One of $A, S, I, J$ is continuous

(c) The pair $(I, A), (J, S)$ are compatible of type (A)

(d) $$F\left\{\begin{matrix} \|Sy - Ix\|^2, \|Sy - Jy\|^2, \|x - Sy\|^2, \|x - Jy\|^2, \|y - Ix\|^2 \\ \|Sy - Ay\|^2 \|Ax - Ix\|^2, \|Ay - Jy\|^2 \end{matrix}\right\} \leq 0 \;\; for\ all\ x, y\ in\ X,$$

$then\ A, S, I$ & $J$ have a unique common fixed point.

**Proof:** Consider $B\ and\ T$ identity mappings in theorem 2.2.

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