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A Subclass of Harmonic Functions Associated withWright's Hypergeometric Functions

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Abstract

We introduce a new class of complex valued harmonic functions associated with Wright hypergeometricfunctions which are orientation preserving and univalent in the open unit disc. Further we define, Wrightgeneralized operator on harmonicfunction and investigate the coefficient bounds, distortion inequalities andextreme points for this generalized class of functions.

Keywords: Harmonic Univalent Starlike Functions, Harmonic Convex Functions, Wright Hypergeometric Functions, Raina-Dziok Operator, Distortion Bounds, Extreme Points

1. Introduction

A continuous function $f = u+iv$is a complex-valued harmonic function in a complex domain G if both u and v are real and harmonic in G. In any simply-connected domain $D \subset G$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [1]). Denote by H the family of functions

$$f = h + \overline{g} \quad (1)$$

which are harmonic, univalent and orientation preserving in the open unit disc U = {z :| z |<1} so that f is normalized by $f(0)=h(0)=f(0)$ -1= 0 f h fz . Thus, for $f = h + \overline{g}$

∈ H, we may express

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m}, \quad |b_1| < 1, \qquad (2)$$

where the analytic functions *h* and *g* are in the forms

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \; g(z) = \sum_{m=1}^{\infty} b_m z^m \quad (|b_1| < 1).$$

We note that the family H of orientation preserving, normalized harmonic univalent functions reduces to the well-known class S of normalized univalent functions if the co-analytic part of f is identically zero, that is g ≡ 0. Let the Hadamard product (or convolution) of two power series

$$\phi(z) = z + \sum_{m=2}^{\infty} \phi_m z^m \qquad (3)$$

and

$$\psi(z) = \sum_{m=2}^{\infty} \psi_m z^m \qquad (4)$$

in *S* be defined by

$$(\phi * \psi)(z) = \sum_{m=2}^{\infty} \phi_m \psi_m z^m \quad .$$

For positive real parameters α_1, A1,...α_p, A_p and $\beta_{1,}$ $B_{1,...}$ β_q B_q(p, q ∈ N = 1, 2, 3, ...) such that

$$1 + \sum_{m=1}^{q} B_m, - \sum_{m=1}^{p} A_m \geq 0 \quad , \qquad z \in U. \qquad (5)$$

The Wright's generalized hypergeometric function [2]

$$p\Psi_q\left[(\alpha_1, A_1), \cdots, (\alpha_p, A_p); (\beta_1, B_1)\cdots(\beta_q, B_q); z\right]$$
$$= p\Psi_q\left[(\alpha_m, A_m)_{1,p}(\beta_m, B_m)_{1,q}; z\right]$$

is defined by

$${}_p\Psi_q\left[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}; z\right]$$
$$= \sum_{m=0}^{\infty}\left\{\prod_{t=0}^{p}\Gamma(\alpha_t + mA_t)\right\}\left\{\prod_{t=0}^{q}\Gamma(\beta_t + mB_t)\right\}^{-1} \cdot$$
$$\frac{z^m}{m!}, \quad z \in U.$$

If $A_t = 1$ $(t = 1, 2, p)$ and $B_t = 1$ $(t = 1, 2, q)$ we have

the relationship:

$$\Omega\, {}_p\Psi_q\left[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}; z\right]$$
$$= {}_pF_q\left(\alpha_1, \cdots \alpha_p; \beta_1 \cdots, \beta_q; z\right)$$
$$= \sum_{m=0}^{\infty}\frac{(\alpha_1)_m, \cdots(\alpha_p)_m}{(\beta_1)_m \cdots, (\beta_q)_m}\frac{z^m}{m!},$$

$(p \leq q+1; p.q \in N\circ = N \bigcup \{0\}; Z \in U)$ is the generalizedhypergeometric function (see for details [3]) where Ndenotes the set of all positive integers and $(\alpha)_n$ is thePochhammer symbol and

$$\Omega = \left\{\prod_{t=0}^{p}(a_t)\right\}^{-1}\left\{\prod_{t=0}^{q}T(\beta_t)\right\} \quad (7)$$

By using the generalized hypergeometric function Dziok and Srivastava [3] introduced the linear operator. In [4] Dziok and Raina extended the linear operator by using Wright generalized hypergeometric function. First we define a function

$$
{}_p\Phi_q\left[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}; z\right]
$$
$$
= \Omega\, {}_p\Psi_q\left[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}; z\right]
$$

Let $W\left[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}\right]: S \to S$ be a linear operator defined by

$$
W\left[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}\right]\varphi(z)
$$
$$
:= z\, {}_p\Phi_q\left[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}; z\right] * \varphi(z)
$$

We observe that, for $f(z)$ of the form (1), we have

$$
W\left[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}\right]\varphi(z)
$$
$$
:= z + \sum_{m=2}^{\infty} \sigma_m(\alpha_1)\,\varphi_m z^n, \tag{8}
$$

where $\sigma_m(\alpha_1)$ is defined by

$$
\sigma_m(\alpha_1)
$$
$$
= \frac{\Omega\Gamma(\alpha_1 + A_1(m-1))\cdots\Gamma(\alpha_p + A_p(m-1))}{(m-1)!\Gamma(\beta_1 + B_1(m-1))\cdots\Gamma(\beta_q + B_q(m-1))} \tag{9}
$$

If, for convenience, we write

$$
W[\alpha_1]\phi(z)
$$
$$
= W\left[(\alpha_1, A_1), \cdots (\alpha_p, A_p); (\beta_1, B_1)\cdots(\beta_q, B_q)\right]\phi(z) \tag{10}
$$

introduced by Dziok and Raina [4].
It is of interest to note that, if $A_t = 1$ (t = 1, 2, ..., p), $B_t = 1$ (t = 1, 2, ...,q) in view of the relationship (6) the linear operator (8) includes the Dziok-Srivastava operator (see [3]), for more details on these operators see [3,4,6,7] and[8]. It is interesting to note that Wright generalized hypergeometric function contains, Dziok-Srivastava operator as its special cases, further other linear operators the Hohlov operator, the Carlson-Shaff er operator [6], the Ruscheweyh derivative operator [7], the generalized Bernardi-Libera-Livingston operator, the fractional derivative operator [8], and so on. For example ifp = 2 and

$q = 1$ with $\alpha_1 = \delta + 1$ $(\delta > -1)$, $\alpha_2 = 1$, $\beta_1 = 1$, then

$$
W_1^2(\delta + 1, 1; 1)\,\phi(z)
$$
$$
= D^{\delta} f(z) = \frac{z}{(1-z)^{\delta+1}} * \phi(z)
$$

is called Ruscheweyh derivative of order δ (δ > −1).
From (8) now we define, Wright generalized hypergeometric harmonic function $f = h + \overline{g}$ of the form (1), as and $W_q^p[\alpha_1]f(z) = W_q^p[\alpha_1]h(z) + \overline{W_q^p[\alpha_1]g(z)}$ (11). on harmonic function. Motivated by the earlier works of [1,5,9-13] on the subject of harmonic functions, we introduce here a new $WS_H([\alpha_1],\gamma)$ of H.
For $0 \le \gamma < 1$, let $WS_H([\alpha_1],\gamma)$ ırlike harmonic functions
f€H of the form (1)such that

equivalently
$$\frac{\partial}{\partial\theta}(\arg W_q^p[\alpha_1]f(z)) > \gamma \qquad (12)$$

$$\operatorname{Re}\left\{\frac{z(W_q^p[\alpha_1]h'(z)) - \overline{z(W_q^p[\alpha_1]g'(z))}}{W_q^p[\alpha_1]h(z) + \overline{W_q^p[\alpha_1]g(z)}}\right\} > \gamma \qquad (13)$$

where $W_q^p[\alpha_1]f(z)$ (11) and $z \in U$.
We also l $WV_H([\alpha_1],\gamma) = WS_H([\alpha_1],\gamma) \cap V_H$ functions with varying arguments introduced by Jahangiri and Silverman [10], consisting of functions f of the form (1) in H for which there
exists a real number φ such that

$$\eta_m + (m-1)\varphi \equiv \pi(\operatorname{mod} 2\pi),\quad \delta_m + (m-1)\varphi \equiv 0,\quad m \ge 2 \qquad (14)$$

WhereI $\eta_m = \arg(a_m)$ and $\delta_m = \arg(b_m)$. :ient coefficient condition for functions f given by (2) to be in the class
$WS_H([\alpha_1],\gamma)$. that this coefficient conditionis necessary also for functions belonging to the classFurther, d $WV_H([\alpha_1],\gamma)$ lts and extremepoints for functions in $WV_H([\alpha_1],\gamma)$

Theorem 1. Let $f = h + \overline{g}$ by (2). If

$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}|a_m| + \frac{m+\gamma}{1-\gamma}|b_m|\right)\cdot \sigma_m(\alpha_1) \le 1 - \frac{1+\gamma}{1-\gamma}b_1 \qquad (15)$$

$0 \le \gamma < 1$, Then $f \in WS_H([\alpha_1],\gamma)$.

Proof. We first show that if the inequality (15) holdsfor the coefficients of $f = h + \overline{g}$ then the required condition (13) is satisfied. Using (11) and (13), we can write

$$\mathrm{Re}\left\{\frac{z(W_q^p[\alpha_1]h'(z))-\overline{z(W_q^p[\alpha_1]g'(z))}}{W_q^p[\alpha_1]h(z)+\overline{W_q^p[\alpha_1]g(z)}}\right\}$$

$$=\mathrm{Re}\frac{A(z)}{B(z)}$$

where

$$A(z)=z(W_q^p[\alpha_1]h'(z))-\overline{z(W_q^p[\alpha_1]g'(z))}$$

and

$$B(z)=W_q^p[\alpha_1]h(z)+\overline{W_q^p[\alpha_1]g(z)}$$

In view of the simple assertion that $\mathrm{Re}(w)\geq\gamma$y ifit is suffici $|1-\gamma+w|\geq|1+\gamma-w|$ That

$$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|\geq 0. \quad (16)$$

Substituting for A(z) and B(z) the appropriate expressions in (16), we get

$$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|$$

$$\geq(2-\gamma)|z|-\sum_{m=2}^{\infty}(m+1-\gamma)\sigma_m(\alpha_1)|a_m|\cdot$$

$$|z|^m-\sum_{m=1}^{\infty}(m-1+\gamma)\sigma_m(\alpha_1)|b_m||z|^m$$

$$-\gamma|z|-\sum_{m=2}^{\infty}(m-1-\gamma)\sigma_m(\alpha_1)|a_m|\cdot$$

$$|z|^m-\sum_{m=2}^{\infty}(m+1+\gamma)\sigma_m(\alpha_1)|b_m||z|^m$$

$$\geq 2(1-\gamma)|z|\left\{1-\sum_{m=2}^{\infty}\frac{m-\gamma}{1-\gamma}\sigma_m(\alpha_1)|a_m|\cdot\right.$$

$$\left.|z|^{m-1}-\sum_{m=2}^{\infty}\frac{m+\gamma}{1-\gamma}\sigma_m(\alpha_1)|b_m||z|^{m-1}\right\}$$

$$\geq 2(1-\gamma)|z|\left\{1-\frac{1+\gamma}{1-\gamma}b_1-\right.$$

$$\left.\left(\sum_{m=2}^{\infty}\left[\frac{m-\gamma}{1-\gamma}\sigma_m(\alpha_1)|a_m|+\frac{m+\gamma}{1-\gamma}\sigma_m(\alpha_1)|b_m|\right]\right)\right\}\geq 0.$$

by virtue of the inequality (15). This implies that $f\in WS_H([\alpha_1],\gamma)$.

Theorem 2. Letbe given by (2) and for, then if and $0\leq\gamma<1$ $f=h+\overline{g}$ $f\in WV_H([\alpha_1],\gamma)$

$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}|a_m|+\frac{m+\gamma}{1-\gamma}|b_m|\right)\sigma_m(\alpha_1)\leq 1-\frac{1+\gamma}{1-\gamma}b_1 \quad (17)$$

Proof. $WV_H([\alpha_1],\gamma) \subset WS_H([\alpha_1],\gamma)$, the necessary part of the theorem.

Assumethat $f \in WV_H([\alpha_1],\gamma)$ of (11) to (13),weobtain

$$\operatorname{Re}\left\{\frac{z(W_q^p[\alpha_1]h'(z)) - \overline{z(W_q^p[\alpha_1]g'(z))}}{W_q^p[\alpha_1]h(z) + \overline{W_q^p[\alpha_1]g(z)}} - \gamma\right\} \geq 0.$$

The above inequality is equivalent to

$$\operatorname{Re}\left\{\frac{z + \sum_{m=2}^{\infty}(m-\gamma)\sigma_m(\alpha_1)|a_m|z^m - \sum_{m=1}^{\infty}(m+\gamma)\sigma_m(\alpha_1)|b_m|\bar{z}^m}{z + \sum_{m=2}^{\infty}\sigma_m(\alpha_1)|a_m|z^m + \sum_{m=1}^{\infty}\sigma_m(\alpha_1)|b_m|\bar{z}^m}\right\}$$

$$= \operatorname{Re}\left\{\frac{(1-\gamma) + \sum_{m=2}^{\infty}(m-\gamma)\sigma_m(\alpha_1)|a_m|z^{m-1} - \frac{\bar{z}}{z}\sum_{m=1}^{\infty}(m+\gamma)\sigma_m(\alpha_1)|b_m|\bar{z}^{m-1}}{1 + \sum_{m=2}^{\infty}\sigma_m(\alpha_1)|a_m|z^{m-1} + \frac{\bar{z}}{z}\sum_{m=1}^{\infty}\sigma_m(\alpha_1)|b_m|\bar{z}^{m-1}}\right\} \geq 0.$$

This condition must hold for all values of z, such thatz| = r < 1. Upon choosing φ according to (14) we musthave (18).

If (17) does not hold, then the numerator in (18) isnegative for r sufficiently close to 1. Therefore, thereexists a point $z_0=r_0$ in (0, 1) for which the quotient in(18) is negative. This contradicts our assump $f \in WV_H([\alpha_1],\gamma)$ us conclude that it is both necessary and sufficient that the coefficient bound inequality

For a compact family, the maximum or minimum of the real part of any continuous linear functional occurs at one of the extreme points of the closed convex hull. Unlike many other classes, $f \in WV_H([\alpha_1],\gamma)$ by necessary and sufficient coefficient conditions, the family WV_H ([α_1],r) is not a convex family.does not hold, then the numerator in (18) is negative for r sufficiently close to 1. Therefore, there exists a point $z_0=r_0$ in (0, 1) for which the quotient in (18) is negative. This contradicts our assumption thatWe thus conclude that it is both necessary and sufficient that the coefficient bound inequality Nevertheless, we may still apply the coefficient characterization of the WV_H ([α_1],r) to determine the extreme points.

(17) holds true when $f \in WV_H([\alpha_1],\gamma)$ sthe proof of Theorem 2.

If we puti $\varphi = \frac{2\pi}{k}$ then Theorem 2 gives thefollowing corollary.

Corollary necessary and sufficient conditio $f = h + \bar{g}$ fying (17) to be starlike is that arg(a_m) =a $\pi - 2(m-1)\pi/k$, k =1, $2\pi - 2(m-1)\pi/k$, 2,3,…..).

3. Distortion Bounds and Extreme Points

In this section we obtain the distortion bounds for thefunctions $f \in WV_H([\alpha_1],\gamma)$ that lead to a covering result for the family $WV_H([\alpha_1],\gamma)$.

Theorem 3. Iftl $f \in WV_H([\alpha_1],\gamma)$

$$f(z) \le (1+|b_1|)r + \frac{1}{\sigma_2(\alpha_1)}\left(\frac{1-\gamma}{2-\gamma} - \frac{1+\gamma}{2-\gamma}|b_1|\right)r^2$$

And

$$f(z) \ge (1-|b_1|)r - \frac{1}{\sigma_2(\alpha_1)}\left(\frac{1-\gamma}{2-\gamma} - \frac{1+\gamma}{2-\gamma}|b_1|\right)r^2.$$

Proof. We will only prove the right-hand inequality ofthe above theorem. The arguments for the left-hand inequality are similar and so we omit it. Let $f \in WV_H$ $([\alpha_1],\gamma)$ Taking the absolute value of f, we obtain

$$|f(z)|$$

$$\le (1+|b_1|)r + \sum_{m=2}^{\infty}(|a_m|+|b_m|)r^m$$

$$\le (1+|b_1|)r + r^2\sum_{m=2}^{\infty}(|a_m|+|b_m|).$$

This implies that

$$|f(z)| \le (1+|b_1|)r + \frac{1}{\sigma_2(\alpha_1)}\left(\frac{1-\gamma}{2-\gamma}\right)\cdot$$

$$\sum_{m=2}^{\infty}\left(\left(\frac{2-\gamma}{1-\gamma}\right)\sigma_2(\alpha_1)|a_m| + \left(\frac{2-\gamma}{1-\gamma}\right)\sigma_2(\alpha_1)|b_m|\right)r^2$$

$$\le (1+|b_1|)r + \frac{1}{\sigma_2(\alpha_1)}\left(\frac{1-\gamma}{2-\gamma}\right)\left[1 - \frac{1+\gamma}{1-\gamma}|b_1|\right]r^2$$

$$\le (1+|b_1|)r + \frac{1}{\sigma_2(\alpha_1)}\left(\frac{1-\gamma}{2-\gamma} - \frac{1+\gamma}{2-\gamma}|b_1|\right)r^2,$$

which establish the desired inequality.As consequences of the above theorem and corollary 1,we state the following corollary.

Corollary 2. Leta $f = h + \overline{g}$ form (2) be sothat Then $f \in WV_H([\alpha_1],\gamma)$

$$\left\{w : |w| < \frac{2\sigma_2(\alpha_1) - 1 - [\sigma_2(\alpha_1)-1]\gamma}{(2-\gamma)\sigma_2(\alpha_1)} - \frac{2\sigma_2(\alpha_1) - 1 - [\sigma_2(\alpha_1)+1]\gamma}{(2-\gamma)\sigma_2(\alpha_1)}b_1\right\} \subset f(U).$$

For a compact fam he real part of any continuous linear functional occurs alone of the extreme points of the closed convex hull. Unlike many other classes, characterized by necessary andsufficient coefficient conditions, the family WV_H $([\alpha_1],r)$ is not a convex family. Nevertheless, we may stillapply the coefficient characterization of the WV_H $([\alpha_1],r)$results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler harmonic function classes (see [10,12,13]). The details involved in the derivations of suchto determine the extreme points.

Theorem 4. The closed convex hull of($WV_H([\alpha_1],\gamma)$clco) is$WV_H([\alpha_1],\gamma)$

$$\left\{ f(z) = z + \sum_{m=2}^{\infty} |a_m| z^m + \overline{\sum_{m=1}^{\infty} |b_m| z^m}, \right.$$

$$\left. : \sum_{m=2}^{\infty} m\big[|a_m| + |b_m|\big] < 1 - b_1 \right\}.$$

$$\frac{(1-\gamma)-(1+\gamma)b_1 - \left(\sum_{m=2}^{\infty} (m-\gamma)\sigma_m(\alpha_1)|a_m| r^{m-1} + (m+\gamma)\sigma_m(\alpha_1)|b_m| r^{m-1} \right)}{1+|b_1| + \left(\sum_{m=2}^{\infty} \sigma_m(\alpha_1)|a_m| + \sum_{m=1}^{\infty} \sigma_m(\alpha_1)|b_m| \right) r^{m-1}} \geq 0.$$

By setting $\lambda_m = \dfrac{1-\gamma}{(m-\gamma)\sigma_m(\alpha_1)}$ d, th $\mu_m = \dfrac{1-\gamma}{(m+\gamma)\sigma_m(\alpha_1)}$, points for are

$$\left\{ z + \lambda_m x z^m + \overline{b_1 z} \right\} \cup \left\{ z + \overline{b_1 z + \mu_m x z^m} \right\} \qquad (19)$$

Where $m \geq 2$ and $|x| = 1 - |b_1|$.

Proof. Any function f in $WV_H([\alpha_1], r)$ be expressed as

$$f(z) = z + \sum_{m=2}^{\infty} |a_m| e^{i\eta_m} z^m + \overline{b_1 z} + \overline{\sum_{m=2}^{\infty} |b_m| e^{i\delta_m} z^m},$$

where the coefficient satisfy the inequality (15). Set

$h_1(z) = z,\quad g_1(z) = b_1 z,\quad h_m(z) = z + \lambda_m e^{i\eta_m} z^m,\quad g_m(z) = b_1 z$
$+ \mu_m e^{i\mu_m} z^m$, for m = 2, 3,… Writing

In particular, put $X_m = \dfrac{|a_m|}{\lambda_m}$, $Y_m = \dfrac{|b_m|}{\mu_m}$ m = 2, 3... and $X_1 = 1 - \sum_{m=2}^{\infty} X_m$;

$Y_1 = 1 - \sum_{m=2}^{\infty} Y_m$; we get

$$f(z) = \sum_{m=1}^{\infty} [X_m h_m(z) + Y_m g_m(z)].$$

We see that extreme points of functions in clco WV_H. $([\alpha_1],\gamma) \subset \{f_m(z)\}$.

To see that mf is not an extremepoint if bothv $|x| \neq 0$ and $|y| \neq 0$,an then also be expressed as a convex linear combinations of functions in clcoW $WV_H([\alpha_1],\gamma)$. ofgenerality, assume Choose □ □ ($|x| \geq |y|$. nough

$$\in < \frac{|x|}{|y|}. \text{ Set } A = 1 + \in \text{ and } B = 1 - \left| \frac{\in x}{y} \right|. \text{ We}$$

then see that botha $t_1(z) = z + \lambda_m A x z^m + \overline{b_1 z + \mu_m B y z^m}$

are inand that clco $WV_H([\alpha_1],\gamma)$ $f_m(z) = \frac{1}{2}\{t_1(z)$

$t_2(z) = z + \lambda_m (2-A) x z^m + \overline{b_1 z + \mu_m (2-B) y z^m}$

Concluding Remarks

The various results presented in this paper would provideinteresting extensions and generalizations of those considered earlier for simpler harmonic function classes (see[10,12,13]). The details involved in the derivations ofsuchspecializations of the results presented in this paperare fairly straight-forward.

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