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Fuzzy Group and Fuzzy Normal Subgroup

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Abstract: This paper deals with the Fuzzy Group and Fuzzy Normal Subgroup.

Keywords: Group, Subgroup, Fuzzy Set, Fuzzy group, Function

Introduction: The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh and subsequently several authors including Zadeh have discussed various aspects of the theory and application of fuzzy sets such as a fuzzy topological spaces, similarity relations and fuzzy orderings, algebraic properties of fuzzy sets, fuzzy measures, probability measures of fuzzy events, fuzzy mathematical programming, fuzzy dynamic programming and decision making on a fuzzy environment.

Let S be a groupoid i.e., a set closed under binary operation (denoted multiplicatively). The fuzzy set A on S is defined to be a function denoted by μ_A called gradation function from S into [0,1].

When A in an ordinary subset of S we take $\mu_A(x) = 1$, when $x \varepsilon A$ and

$\mu_A(x) = 0$, $x \varepsilon A$.

<u>Definition</u> Fuzzy subgroupoid: The fuzzy set A on a groupoid S is called fuzzy subgroupoid, if

$\mu_A(x, y) \geq \mu_A$ for allx, $y \varepsilon$ S.

Obviously, when A is an ordinary subset of S,

Then $\mu A(x) = 1$ and $\mu A(y) = 1$ when x, $y \varepsilon$ A then from above

$x, y \varepsilon A$

Therefore, $x y \varepsilon A$ when $x \varepsilon A, y \varepsilon A$

A is ordinary subgroupoid of S.

Left Ideal: A subset T of a groupoid S is called left ideal of S if $y \varepsilon T \rightarrow xy \varepsilon T$ for all $x\varepsilon$ S.

Right Ideal: A subset T of a groupoid S is called right ideal of S if $y \varepsilon T \rightarrow xy \varepsilon T$ for all $x\varepsilon$ S.

Fuzzy left Ideal: The fuzzy set A on groupoid S is called fuzzy left ideal if

$\mu_A(x y) \geq \mu_A(y)$ for all $x, y\varepsilon S$.

Fuzzy Right Ideal: The fuzzy set A on groupoid S is called fuzzy right if

$\mu_A(x, y) \geq \mu_A(x) \forall x, y \varepsilon S$

Fuzzy Right Ideal: The fuzzy set A on groupoid S is called fuzzy ideal if it is both left and right ideal.

Obviously, (i) every fuzzy ideal on S is a fuzzy subgroupoid but not conversely because, if A, a fuzzy ideal then it is both left and right ideal.

Hence, $\mu_A(x,y) \geq \mu_A(y)$ for all $x, y\varepsilon$ S.

And, $\mu_A(x,y) \geq \mu_A(x)$ for all $x, y\varepsilon$ S.

Hence, $\mu_A(x,y) \geq \mu_A(x)\ \mu_A(y)$ for all $x, y\varepsilon$ S.

as $\mu_A(x), \mu_A(y)\ \varepsilon\ [0,1]$

Therefore,A is a fuzzy subgroupoid.

Converse, obviously does not hold.

(ii) $\mu_A(x^n) \geq \mu_A{}^n(x)$

(iii) Fuzzy null set ϕ and the fuzzy set S are fuzzy groupoid.

If μ_T be the characteristic function of the subset T of the groupoid S, then fuzzy set defined μ_T is fuzzy subgroupoid or (left, right) ideal if an only if T is a subgroupoid or (left,right) idean resp.

Proof:Suppose μ_T is characteristic function of T c S.

Then, $\mu_T(xy) \geq \mu_T(x)\, \mu_T(y)$

is equivalent to $\mu_T(x) = \mu_T(y) = 1$ implies $\mu_T(xy) = 1$ i.e., x,y in T implies xy in T.
Hence fuzzy set defined by the characteristic function μ_T is fuzzy subgroupoid if and only if T is subgroupoid.
Similarly, $\mu_T(xy) \geq \mu_T(y)$ is equivalent to y in T implies xy in T.
i.e., fuzzy set defined by μ_T is fuzzy left ideal if and only if T is left ideal of S.
If A is fuzzy (left, right) ideal on S then set
$B = \{z: \mu_A(z) \geq \theta\}$ and $\theta \varepsilon [0,1]$ is (left, right) ideal of S.
Proof: Let S be a groupoid and A be a fuzzy left ideal on S and
$B = \{z: \mu_A(z) \geq \theta\}$, $\theta \varepsilon [0,1]$
Let $y\theta B$ and $x\varepsilon$ S
Since A is a fuzzy left ideal
$\mu_A(x, y) \geq \mu_A(y) \geq \theta$
Hence xy θB if y θB.Therefore, B is the left ideal. Similar is proof for right ideal.
Intersection of any set of fuzzy subgroupoids, (left, right) ideals is a fuzzy subgroupoid, (left, right) ideals.
Proof Suppose $\{\mathbf{A}_i\}$, i ε I is any set of fuzzy sub-groupoid of S.
Now for x, yε S
$\mu_{\cap_i \mathbf{A}_i}(x, y) = \inf \mu_{A_i}(x\, y)$
$\geq \inf_i(\mu_{A_i}(x)\, \mu_{A_i}(y))$
$= (\inf \mu_{A_i}(x))\, (\inf \mu_{A_i}(y))$
$= \mu_{\cap_i \mathbf{A}_i}(x)\, \mu_{\cap_i \mathbf{A}_i}(y)$
Therefore, $\cap \mathbf{A}_i$ is a fuzzy subgroupoid.
Suppose $\{\mathbf{A}_i\}$, $i\varepsilon$ I is any set of fuzzy left ideal.
Now, $\mu_{\cap \mathbf{A}_i}(xy) = \text{in}\{\mu_{A_i}(xy)\} \geq \inf_i \mu_{A_i}(y) = \mu_{\cap_i \mathbf{A}_i}(y)$
Hence $\cap \mathbf{A}_i$ is left ideal
Similar is the proof for right ideal.
Image and pre-image of a fuzzy set under a mapping
Definition: If f:X→ y and A be a fuzzy set on X. Then fuzzy set B defined on Y as
$\mu_B(y) = \text{Sup}\, \mu_A(x), y\varepsilon$ Y
$x\varepsilon$ $f^{-1}(y)$ is called image under f of fuzzy set A in X.
Definition: If B is the fuzzy set in Y. Then the fuzzy set A on X defined by
$\mu_A(x) = \mu_B(f(x))$ for all$x\varepsilon$ X.
is called pre-image of B under f.

If f be a mapping from a groupoid X into a groupoid Y such that
$f(x_1x_2) = f(x_1)(x_2)$ for all $x_1x_2\, \varepsilon X$
and B be a fuzzy subgroupoid or (left, right) ideal on Y, then pre-image of B under f is also a fuzzy groupoid or (left,right) ideal on X.
Proof Let X,Y be groupoid and f:X → Y
such that $f(x_1x_2) = f(x_1)f(x_2)$ for all $x_1x_2\, \varepsilon X$.
If fuzzy set A on X be pre-image of fuzzy groupoid B on Y, then,
$\mu_A(x_1x_2) = \mu_B(f(x_1x_2))x_1, x_2 \varepsilon X$
$= \mu_B(f(x_1)f(x_2))$
$\geq \mu_B(f(x_1))\, \mu_B(f(x_2))$

$= \mu_A(x_1)\ \mu_A(x_2)$ for all $x_1x_2\ \varepsilon X$

Therefore, A is a fuzzy groupoid on X.

Similar is the proof for left and right ideals.

If f is one-one mapping of a groupoid X onto a groupoid Y such that $f(x_1x_2)=f(x_1)f(x_2)$ for all $x_1x_2\ \varepsilon X$ and A be a fuzzy subgroupoid or (left,right) ideal on X, then image fuzzy set B of A under f is a fuzzy groupoid or (left, right) ideal on Y.

Proof: If $x_{1,}x_2 \varepsilon X$ we have

$\mu_B(f(x_1)f(x_2))=\text{Sup}\ \mu_A(z)$

$z\varepsilon f^{-1}(f(x_1)f(x_2))$

$= \mu_A(x_1x_2)$

$\geq \mu_A(x_1)\ \mu_A(x_2)$

$= \mu_B(f(x_1))\ \mu_B(f(x_2))$

Similar is the proof for (left, right) ideal.

Fuzzy Group and Fuzzy Subgroup

Definition: If S is a group, then fuzzy set A on S is called a fuzzy subgroup of S if

(i)$\mu_A(x_, y) \geq \mu_A(x)\ \mu_A(y)$ for all x, y ε S

(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ for all x ε S

Proposition: If A is a fuzzy subgroupoid of a group S, Then

(i) $\mu_A(x_, y)= \mu_A(x)$ for x ε S

(ii) $\mu_A(e) \geq \mu^2{}_A(x)$, e is the identity element of S.

(iii) $\mu_A(e) \leq 1$, e is the identity element of S.

(iv) $\mu_A(e) = 1$ of $\mu_A(x) = 1$ for at least on .

Proof(i) We have

$(\mu_A(x^{-1})^{-1}) \geq \mu_A(x^{-1})$

i.e. $\mu_A(x) \geq \mu_A(x^{-1})$

and also $\mu_A(x^{-1}) \geq \mu_A(x)$ for all x ε S

Therefore, $\mu_A(x\, x^{-1}) \geq \mu_A(x^{-1})$ for all x ε S

(ii) $\mu_A(x\, x^{-1}) \geq \mu_A(x)\ \mu_A(x^{-1})$

$\rightarrow \mu_A(e) \geq \mu^2{}_A(x)$ for all x ε S

(iii)We know that

$\rightarrow \mu_A(ex) \geq \mu_A(e)\ \mu_A(x)$e is the identity element of S

$\rightarrow \mu_A(x) \geq \mu_A(e)\ \mu_A(x)$

$\rightarrow \mu_A(e) \leq 1$

(iv)Obvious from (ii) and (iii)

If T is a subgroup of a group S, then the fuzzy set defined by the characteristic function μ_T is a fuzzy subgroup of S and conversely.

Proof: If, then x,y$\varepsilon\ T$, then $\mu_T(x) = 1$, $\mu_T(y) = 1$. If T is a subgroup, then

x y ε and $\mu_T(xy) = 1$

therefore, $\mu_T(xy) \geq \mu_T(x)\ \mu_T(y)$ holds

if x ε T and y ε T, then $\mu_T(x)=1$ and $\mu_T(y)=0$

therefore, $\mu_T(x)\ \mu_T(y)=0$

i.e. $\mu_T(xy) \geq \mu_T(x)\ \mu_T(y)$ holds.

Again if x $\notin$T , $x^{-1} \notin T$, then $\mu_T(z^{-1})= \mu_T=1$ holds.

If x $\varepsilon\ T$, $x^{-1}\varepsilon\ T$ and then $\mu T(z^{-1}) = \mu_T =0$ holds.

Hence ,μ_T is a fuzzy subgroup of S. Conversely, suppose μ_T is the characteristic function of $T \leq S$ and the fuzzy set defined by μ_T is a fuzzy sub-group of S because, if x,y$\varepsilon\ T$, then

$\mu_T(x)= \mu_T(y)=1$

But $\mu_T(xy) \geq \mu_T(x)\ \mu_T(y)=1$

Hence μ_T=(xy)=1
Therefore, x y ε T ………….(1)
If x ε T then μ_T(x)=1
Since $\mu_T(x^{-1})= \mu_T(x)=1$
Therefore, x^{-1} εT
Hence, This a subgroup of S.
The intersection of any set of fuzzy subgroups of a group S is a fuzzy subgroup of S.
Proof: Let $\{A_i\}, i\varepsilon T$ be a fuzzy subgroups of S. By prop 4.3, $\cap_{A_i}$is a fuzzy subgroupoid of S.
Again if x ε S
$\mu \cap_{A_i} (x^{-1}) = \inf_{i\varepsilon I} \mu_{a_i}(x^{-1}) \geq \inf \mu_{A_i}(x) = \mu_{A_i}(x)$
Hence, $\cap_{A_i}$ is a fuzzy subgroup of S.
If f is a homomorphi m from group X onto a group Y, then the pre-image under f or fuzzy subgroup B in Y is fuzzy subgroup in X.
Proof : Let A be the pre-image of B. Then, from prop. 4.4, we have
$\mu_A(x_1x_2) \geq \mu_A(x_1)\ \mu_A(x_2), \forall\ x_1x_2\ \varepsilon\ X$
And$\mu_A(x^{-1}) = \mu_B(f(x^{-1}))$
$= \mu_B (f(x^{-1}))^{-1}$
$\geq \mu_B f(x)$
$= \mu_A (x)$
Therefore, A is fuzzy subgroup of X.
If f is an isomorphism of group X onto group Y and A is a fuzzy subgroup of X, then image B of A under f is a fuzzy subgroup of Y.
Proof If $x_1,x_2\varepsilon X$ then $f(x_1),f(x_2)\ \varepsilon$ y
Now from prop. 4.5, we have
$\mu_B (f(x_1)\ f(x_2)) \geq \mu_B f(x)\ \mu_B (f(x_2))$
And, $\mu_B (f(x_1))^{-1} = \mu_B (f(x^{-1}))$
$= \mu_A (z)$
$Z\ \varepsilon\ f^{-1} (f (x^{-1}))$
$= \mu_A (x^{-1}) \geq \mu_A (x)$
$= \mu_B(f(x))$
Hence, B is fuzzy subgroup of Y.

Conclusion: The fuzzy group and fuzzy subgroup has a verified relevance in the mode of algebra. The use of fuzzy set in algebra is of utmost importance and the relevance has been explicitly explained above.

References

(1) J.M. Annftory and H. Sherwood, Fuzzy groups redefined, J. Math. Anal. Appl. 69 (1979) 124-130.

(2) J.M. Annftory and H. Sherwood, A characterisation of fuzzy subgroups, Fuzzy Sets and system 7 (1982) 279-305.

(3) D.H. Faser, fuzzy topological groups, J. Math. Anal. Appl. 67 (1979) 549-564.

(4) A.K. Kausaras and D.B. Liu, Fuzzy vector spaces and fuzzy topological vector spaces, J. Math Anal. Appl. 58 (1977) 135-146.

(5) Wang-Jim Liu Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and System 8 (1982) 133-139.

(6) Wang-Jim Liu, Operations on fuzzy ideals, Fuzzy Sets and Systems 11 (1982) 31-41.

(7) M.T. Abu osman, On the direct product of fuzzy subgroups, Fuzzy Sets and Systems 12 (1984) 87-91.

(8) A. Rosemfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512-517.

(9) Wang-miingwu, normal fuzzy subgroups, fuzzy math. 1 (1981) 512-21-30.

(10) L.A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338-353.