International Research Journal of Natural and Applied Sciences
ISSN: (2349-4077)
Impact Factor 5.46 Volume 7, Issue 01, January 2020
Website- www.aarf.asia, Email : editoraarf@gmail.com

A COMMON FIXED POINT THEOREM FOR FOUR SELF MAPPINGS IN FUZZY METRIC SPACE

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ABSTRACT

The purpose of this paper is to establish a common fixed point theorems in fuzzy metric space using the concept of weak compatible mappings for four self-mapping and generalizing the result of Sharma et. al.[8]. We also cite an example in support of ourresult.

Keywords: Common fixed point, Fuzzy metric space, R-weakly commuting maps, compatible mappings, weak compatiblemappings.

AMS (2000) Subject Classification: 54H25, 47H10.

INTRODUCTION

The concept of fuzzy set was introduced by Zadeh (1965) [12] as a new way to represent vagueness in everyday life. A large number of renowned Mathematicians worked with fuzzy sets in different branches of Mathematics. One such is the Fuzzy Metric Space. In this paper we use the concept of fuzzy metric space introduced by Kramosil and Michalek [4] and modified by George and Veeramani [1] with the help of t-norm. Grabiec [5] obtained the fuzzy version of Banach contraction principle, which is a milestone in developing fixed point theory in fuzzy metric space. In the sequel, Vasuki [7] introduced the concept of R-weakly commuting in fuzzy metric space and proved the common fixed point theorem. The concept of compatibility was generalized by Jungck [3] .The concept of compatibility in fuzzy metric space was proposed by Mishra *et. al.*, [6]. In 1996 Jungck again generalized the notion of compatible mapping by introducing weak mapping [2]. In 2005, Singh and Jain [10] introduced the concept of semi-compatible mappings and proved the common fixed point theorem for four self-mapping in fuzzy metric space. That was generalization of Vasuki[7].

In 2012,Sharma *et. al.*, [8] also generalized the result for three commuting mappings instead of two mapping, given by Vasuki [7].

In this paper we generalize the result for four weak compatible mappings instead of three mapping, given by Sharma *et. al.*, [8]. Our result is also a generalization of Singh and Jain [10].

1. PRELIMINARIES

Definition 2.1. [12] Let X be any set. A fuzzy set A in X is a function with domain in X and values in [0, 1].
Definition 2.2. [6] Abinaryoperation $*: [0,1] \times [0,1] \to [0,1]$ ed a continuous t-norm if it satisfies the followingconditions:

(i) $*$ is associative andcommutative,

(ii) $*$ iscontinuous,

(iii) $a * 1 = a$, for all $a \in [0,1]$,

(iv) $a * b \le c * d$, whenever $a \le c$ and $b \le d$ for all a, b, c, d $\in$ [0, 1].

Examples of t-normsare

$a * b = \min\{a, b\}$ (minimum

t-norm), $a * b = ab$ (productt-

norm).

Definition2.3. [1] The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is anarbitraryset, $*$ is acontinuous t- norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the followingconditions:

(FM-1) M(x, y, t) > 0,

(FM-2) M(x, y, t) = 1 if and only

if x = y, (FM-3) M(x, y, t) =

M(y, x, t),

(FM-4) $M(x,y,t) * M(y, z, s) \le M(x, z, t+s)$,

(FM-5) M(x, y, .) : $(0, \infty) \to [0, 1]$ is continuous,

for all x, y, z $\in$ X and t, s > 0.

Let(X,d)beametricspaceandlet $a * b = ab$ or $a * b = \min\{a, b\}$ foralla,b $\in$ [0,1].

$$M(x, y, t) = \frac{t}{t + d(x,y)} \quad ;\text{for all } x, y \in X \text{ and } t > 0.$$

Then $(X, M, *)$ is a fuzzy metric space, and this fuzzy metric M induced by d is called the standard fuzzy metric [1].

Definition2.4.[6]Asequence $\{x_n\}$ inafuzzymetricspace $(X, M, *)$ issaidtobeconvergenttoapoint $x \in X$,if $\lim_{n\to\infty} M(x_n, x, t) = 1$ forall t >0.

Further, the sequence $\{x_n\}$ is said to be Cauchyif

$\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all t > 0 and p >0.

The space $(X,M,*)$ is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 2.5. [5] Let $(X,M,*)$ be a fuzzy metric space. Then M is non-decreasing for all $x,y \in X$.

Lemma2.6.[6]Let $(X,M,*)$ beafuzzymetricspace.ThenMisacontinuous function on $X^2(0,\infty)$.

Throughoutthispaper $(X,M,*)$ willdenotethefuzzy metricspacewiththefollowingcondition:

(FM-6) $\lim_{n\to\infty} M(x,y,t) = 1$ for all $x,y \in X$ and t >0.

Definition 2.7. [7] Two mappingsfandgof a fuzzy metric space $(X,M,*)$ into itself are

$M(fgx,gfx,t) \geq M(fx,gx,t) \quad for\ each\ x\ in\ X.$

said to be weakly commutingif

Definition2.8.[8]Twomappings f and ofafuzzymetricspace $(X,M,*)$ into

itselfaresaid to be R-weakly commutingif

$$M(fgx,gfx,t) \geq M\left(fx,gx,\frac{t}{R}\right) \quad for\ all\ x\ in\ X$$

Remark: R-weakly commutativity implies weak commutativity only whenR≤ 1.

Definition 2.9. [9] Let f and be self-mappings on a fuzzy metric space $(X,M,*)$. The pair (f,g) is said to compatibleif

$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ for all t > 0,
Whenever $\{x_n\}$ is a sequence in Xsuchthat $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ $for\ some\ z \in X.$

Definition 2.10.[11]Let f and be self-mappings on a fuzzy metric space $(X,M,*)$. Then the mappings are said to be weakly compatible if they commute at their coincidencepoint,

thatis, $fx = gx implies\ fgx = gfx.$
It is known that a pair of (f,g) compatible maps is weakly compatible but converse is not true ingeneral.

Definition 2.11. [10]A pair (A, B) of self maps of a fuzzy metric space (X, M, *) is said to be semi-compatible if $\lim_{n\to\infty} ABx_n = Bx$,whenever $\{x_n\}$ isasequence inX suchthat

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x.$$

It follows that if (A,B) is semi-compatible and Ax = Bx then ABx = BAx that means every semi-compatible pair of self-maps is weak compatible but the converse is not true in general.

In 2012,Sharma et.al.[8]proved the following result:

Theorem 2.12. Let f, g and h be three self-mappings on a fuzzy metric space $(X, M, *)$ satisfying following (2.11.1) & (2.11.2) conditions. Suppose that h is continuous and pairs (f, h) & (g, h) are R-weakly commuting on X. Then f, g and h have a unique common fixed point in X.

(2.12.1) $f(X) \cap g(X) \subset h(X)$.

(2.12.2) $M(fx, gy, t) \geq r\{M(hx, hy, t)\}$ $for\ all\ x, y\ in\ X$.
Wherer : [0,1] →[0,1] is a continuous function such that r(t) > t for each 0 < t <1.

3. MAIN RESULT

Our result generalizes the results of Sharma et.al.[8] as we are using the concept of weak compatibility, which are lighter conditions than R-weakly commuting, compatibility and semi compatibility. We are proving the result for four self-mapings in a fuzzy metric space such that only one weak compatible pair issufficient.

Theorem 3.1: Let $(X, M, *)$ be a complete fuzzy metric space and let P, Q, A, B and be mappings from X into itself such that the following conditions aresatisfied

(3.1.1) $P(X) \cap Q(X) \subset AB(X)$,
(3.1.2) (P, AB) or (Q, AB) isweak-compatible,

(3.1.3) $AB = BA$, $PB = BP$.

(3.1.4)Forevery $x, y\ in\ X\ and\ t > 0$,

$M(Px, Qy, t) \geq rM(ABx, ABy, t)$

Where r : [0,1] →[0,1] is a continuous function such that r (t) > t

for each 0 < t < 1.

Then P, Q, A and B have a unique common fixed point inX.

Proof. Let x_0 be an arbitrary point inX.

According to (3.1.1), there existssome points $x_1, x_2 \in X$

suchthat

$Px_0 = ABx_1 = y_0$ and $Qx_1 = ABx_2 = y_1$.

We canconstruct sequences $\{x_n\}$ and $\{y_n\}$ in X suchthat

$y_{2n} = Px_{2n} = ABx_{2n+1}$ and $y_{2n+1} = Qx_{2n+1} = ABx_{2n+2}$ for n=0,1,2,…….

Now, we first show that $\{y_n\}$ is a Cauchy sequence in X.

Using condition (3.1.4)

$M(y_{2n+1}, y_{2n}, t) = M(Px_{2n}, Qx_{2n+1}, t)$

$M(y_{2n+1}, y_{2n}, t) \geq rM(ABx_{2n}, ABx_{2n+1}, t),$

$\geq rM(y_{2n-1}, y_{2n}, t)$

$M(y_{2n+1}, y_{2n}, t) \geq M(y_{2n-1}, y_{2n}, t) \quad \ldots..(i)$

Similarly, $M(y_{2n+2}, y_{2n+1}, t) \geq M(y_{2n}, y_{2n+1}, t) \quad \ldots..(ii)$

From (i) and (ii)

Therefore $\{M(y_{n+1}, y_n, t)\}$ is an increasing sequence of positive real numbers in [0, 1] and tends to limit l≤1. Now we prove that l =1.

Let we suppose that l<1then, $M(y_{n+1}, y_n, t) \geq r\, M(y_n, y_{n-1}, t)$.

On taking n we get

$\lim_{n\to\infty} M(y_{n+1}, y_n, t) \geq r \lim_{n\to\infty} M(y_n, y_{n-1}, t).$

Now for any positive integer p,

$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \ldots \ldots \ldots * M(y_{n+p-1}, y_{n+p}, t/p).$

Hence, thesubsequences$\{Px_{2n}\}$, $\{Qx_{2n+1}\}$,{AB }and$\{AB(x_{2n+2})\}$ also convergetoz.

$\lim_{n\to\infty} Px_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} ABx_{2n+1} = \lim_{n\to\infty} ABx_{2n+2} = z.$

$x_{2n} \qquad x_{2n+1}$

Case (1) When $(P, AB)$is weak compatiblemappings,

$\lim_{n\to\infty} Px_{2n} = z \quad and \quad \lim_{n\to\infty} ABx_{2n+1} = z,$

Thenthereexist $u \in X$suchthat$ABu = z$.

Step(I)Putting $x = u$ and $y = x_{2n+1}$in condition[3.1.4]

$M(Pu, Qx_{2n+1}, t) \geq rM(ABu, ABx_{2n+1}, t)$

$M(Pu, Qx_{2n+1}, t) \geq rM(z, ABx_{2n+1}, t)$

Let n and using above result we get

$M(Pu, z, t) \geq rM(z, z, t > M(z, z, t)$

$M(Pu, z, t) \to 1 \; as \; n \to \infty$

$Pu = z$

$Pu = ABu = z.$

This result so that 'u' is coincident point of X such that $Puz = ABu =$

$Pu = ABu = z$ and (P, AB) weak compatiblemappings

$PABu = ABPu$.

$ABu = z \rightarrow PABu = Pz$.

$Pu = z \rightarrow ABPu = ABz$

$Pz = ABz$. ……(iii)

Putting $x = z$ $and\ y = x_{2n+1}$ in[3.1.4]

$M(Pz, Qx_{2n+1}, t) \geq rM(ABz, ABx_{2n+1}, t)$

$M(Pz, Qx_{2n+1}, t) \geq rM(Pz, ABx_{2n+1}, t)$

Letting n □ □ □ □ and using above result we get

$M(Pz, z, t) \geq rM(Pz, z, t)$.

$\Rightarrow M(Pz, z, t) > M(Pz, z, t)$ which is contradiction

Hence, $Pz = z$.

Using (iii)weget $ABz = Pz = z$. ….(iv)

Now again putting $x = x_{2n}$ $and\ y = z$ in[3.1.4]

$M(Px_{2n}, Qz, t) \geq rM(ABx_{2n}, ABz, t)$

$M(Px_{2n}, Qz, t) \geq rM(ABx_{2n}, z, t)$

Letting n □ □ □ □ and using above result we get

$M(z, Qz, t) \geq rM(z, z, t)$

$> M(z, z, t)$

$Qz = z$. …..(v)

$Pz = Qz = z = ABz$ since (iii), (iv) and(v)

Case (2). When (Q, AB) is weak compatiblemappings,

$\lim_{n\to\infty} Qx_{2n+1} = z$ and $\lim_{n\to\infty} ABx_{2n} = z$, Then there exist $v \in X$ such that $ABv = z$.

Putting $x = x_{2n}$ and $y = v$ in condition [3.1.4]

$M(Px_{2n}, Qv, t) \geq rM(ABx_{2n}, ABv, t)$

$M(Px_{2n}, Qv, t) \geq rM(ABx_{2n}, z, t)$

Letting n and using above result we get

$M(z, Qv, t) \geq rM(z, z, t)$

$M(z, Qv, t) \geq r(1)$

$Qv = z$

$Qv = ABv = z.$

$'v'$ is a coincident point of X such that $Qv = ABv = z$

Now, $Qv = ABv = z$ and $\{Q, AB\}$ weak compatible mappings.

$QABv = ABQv$

$ABv = z \rightarrow QABv = Qz.$

$Qv = z \rightarrow ABQv = ABz$

$Qz = ABz$.……..(vi)

Now again putting $x = x_{2n}$ $and\ y = z$ in [3.1.4]

$M(Px_{2n}, Qz, t) \geq rM(ABx_{2n}, ABz, t)$

$M(Px_{2n}, Qz, t) \geq rM(ABx_{2n}, Qz, t)$,since (vi)

Letting n$\rightarrow \infty$and using above result we get

$M(z, Qz, t) \geq rM(z, Qz, t)$

$M(z, Qz, t) > M(z, Qz, t)$, which is contradiction

Hence $Qz = z.$

$Qz = ABz = z.$

Putting $x = z$ and $y = x_{2n+1}$ in [3.1.4]

$M(Pz, Qx_{2n+1}, t) \geq rM(ABz, ABx_{2n+1}, t)$

$M(Pz, Qx_{2n+1}, t) \geq rM(z, ABx_{2n+1}, t).$

Letting $n \to \infty$ and using above result we get

$M(Pz, z, t) \geq rM(z, z, t)$

$\Rightarrow M(Pz, z, t) > r(1)$, which is contradiction

Hence, $Pz = z$. $Pz = Qz = z = ABz$

Now, according to case (1) and case (2) we get $Pz = Qz = z = ABz$ .....(vii)

Putting $x = Bz$ and $y = x_{2n+1}$ in [3.1.4]

$M(PBz, Qx_{2n+1}, t) \geq rM(AB(Bz), ABx_{2n+1}, t)$

$M(BPz, Qx_{2n+1}, t) \geq rM(BABz, ABx_{2n+1}, t)$, since (3.1.3)

$M(Bz, Qx_{2n+1}, t) \geq rM(Bz, ABx_{2n+1}, t)$, Since(vii)

Letting $n \to \infty$ and using above result weget

$M(Bz, z, t) \geq rM(Bz, z, t).$

$Bz = z$

$Bz = z$ and $ABz = z$

$Az = z$

Thus, $Pz = Qz = Az = Bz = z.$

z' is a common fixed point of mapping P, Q, A and B.

Uniqueness:

Let 'w' be another common fixed point of P, Q, A and B.

Then $Pz = Qz = Az = Bz = z$ and $Pw = Qw = Aw = Bw = w.$

Putting $x = z$ and $y = w$ in [3.1.4]

$M(Pz, Qw, t) \geq rM(ABz, ABw, t)$

$M(z, w, t) \geq rM(Az, Aw, t)$

$M(z, w, t) \geq rM(z, w, t)$

$z = w.$

Therefore 'z' is unique common fixed pointof P, Q, A and X mappingsof .

Remark 3.2. If we take B = I (identity mapping)in theorem 3.1 then the condition (3.1.3) is satisfied trivially and we get the followingresult.

Corollary 3.3: Let P, Q and A be self-mappings of complete fuzzy metric

space $(X, M, *)$. Suppose that the following conditions aresatisfied:

(3.1.1) $P(X) \cap Q(X) \subset A(X)$,

(3.1.2) Pair (P, A) or (Q, A) is

weakcompatible,(3.1.3)

$For\ every\ x, y\ in\ X\ and\ t > 0$,

$M(Px, Qy, t) \geq rM(Ax, Ay, t)$.

Where r: [0,1] →[0,1] is a continuous function such that r (t) > t for each 0 < t < 1.

Then P, Q and A have a unique common fixed point in X.

Example 3.4: Let $X = [0, 3]$ and (X, d) be a metric space where metric $d(x, y) = |x - y|$.

Define $a * b = a.b$ $a, b \in [0,1]$ and M fuzzy set on $X^2 \times (0, \infty)$ such that

$$M(x, y, t) = \frac{|x - y|}{t + |x - y|}$$

We take three functions $f, g, h : [0,3] \to [0, 3]$ define as follow:

$$f(x) = \begin{cases} x & x \in [0, 1) \\ 3 & x \in [1, 3] \end{cases}, g(x) = 3\ for\ all\ x \in [0,3] \quad \text{and} \quad h(x) = \begin{cases} 3 - x & x \in [0, 1) \\ 3 & x \in [1, 3] \end{cases}.$$

Then $f(X) = [0, 1) \cup \{3\}, g(X) = \{3\}$ and $h(X) = (2, 3]$.

Therefore, $f(X) \cap g(X) \subseteq h(X)$

When $x \in [0, 1]$ $f(x) = x$ and $h(x) = 3 - x$

$fh(x) = f(3 - x) = 3$ and $hf(x) = h(x) = 3 - x$.

$$M(f(x), hx, t) = \frac{|x - (3 - x)|}{t + |x - (3 - x)|} = \frac{|2x - 3|}{t + |2x - 3|}$$

$$M(fhx, hfx, t) = \frac{|3-(3-x)|}{t+|3-(3-x)|} = \frac{|x|}{t+|x|}$$

When putting $x = \frac{1}{2} \in [0,1]$

$$M\left(f\left(\frac{1}{2}\right), h\left(\frac{1}{2}\right), t\right) = \frac{2}{t+2}$$

$$M\left(f\left(\frac{1}{2}\right), h\left(\frac{1}{2}\right), t/2\right) = \frac{2}{(t/2)+2} = \frac{4}{t+4}$$

And

$$M\left(fh\left(\frac{1}{2}\right), hf\left(\frac{1}{2}\right), t\right) = \frac{1/2}{t+1/2} = \frac{1}{2t+1}$$

Hence, we get $M\left(f\left(\frac{1}{2}\right), h\left(\frac{1}{2}\right), t/2\right) > M(fh(1/2), hf(1/2), t)$.

So that $M(fhx, hf(x), t/2) \geq M(fx, hx, t)$, is not satisfiedforall $x \in [0,1]$.

Hence, (f,h) is not R-Weakly commuting mapping but (f,h) is

weakcompatible mapping. Because $f(x) = h(x)$ $when$ $x \in [1,3]$

Then there exist a coincidentpoint x in $[0,3]$ suchthat $fh(x) = hf(x)$ $when$ $x \in [1,3]$

(f,h) is commuting at coincident points.

Similarly (g,h) is also weak compatible $g(x) = h(x)$ $when$ $x \in [1,3]$

$g(x) = h(x) \Rightarrow gh(x) = hg(x)$ $when$ $x \in [1,3]$.

$g(3) = h(3) = 3 \Rightarrow gh(3) = hg(3)$.

Then exist a common unique fixed point 3 in X such that $f(3) = g(3) = h(3) = 3$.

Remark:

Example (3.3) shows that (f,h) is not R-Weakly commuting mapping but there exist a unique common due to the weak compatibility. Every R-Weakly commuting mapping is weak compatible but converse is nottrue.

CONCLUSION

This paper is generalization of the result of Sharma et.al. [8] in the sense of replace R-Weakly commuting to weakly compatible (owc) to prove a theorem on common fixed point theorems for fourself mappings in complete fuzzy metricsspace.

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