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A Study on Contribution to fundamental and homotopy theory

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Abstract.

The fundamental group is defined using loops in topological spaces, which is the first of a series of invariants called homotopy groups.This research provides a treatment of Homological Algebra which approaches the subject in terms of its origins in algebraic topology. The basic goal is to find algebraic invariants that classifytopological spaces up to homeomorphism, though usually most classify up to homotopy equivalence.

Keywords: Topological spaces, Homotopy groups, Homological Algebra.

1. Homotopy Theory

Let X and Y be two topological spaces and f, g : X →Y be two continuous maps. Then f is said to be homotopic to g denoted by f ≃g if there exists a continuous map F : X x I →Y such that F{x, 0) = f(x) and F{x, 1) = g(x), z €X.

We write F :f ≃g to represent c homotopy from f to g.

Two mappings f and g of a space X into a space Y are homotopic (and we write f $\simeq$ g) if there is a mapping $h{:}X \times I^1 \rightarrow Y$ such that for each point x in X,

$$h(x,0) = f(x) \qquad \text{and} \qquad h(x,1) = g(x).$$

This is just another way of that $h|X \times 0 = f \textit{ and } h|X \times 1 = g,$ and hence we have the connection with 1-parameter families. The mapping h is called a homotopy between f and g and the product space $X \times I^1$ is the homotopy cylinder.

In these terms, the mapping $h|X^1 \times 0$ and $h|X^1 \times 1$ shown in Fig. 1.1 are homotopic mapping of S^1 into E^2. Any mapping of S^1 into E^2 is homotopic to any other such mapping, so our example is rather trivial. Such a statement is not true for every space Y, of course. For instance, let Y be the punctured plane $E^2 \rightarrow (0,0)$. Then a constant mapping c of S^1 onto a single point p of cannot be homotopic to a mapping of f of s^1 onto a simple closed curve *J* passing around the (missing) orign (see Fig.1.2). Intuitively, it is impossible to deform J continuously onto the point p while remaining in the space Y.

The question of the existence of a homotopy between two mappings f, g:*X*→*Y* can be very difficult. The answer depends upon *f* and *g*, certainly, and also upon the structure of the spaces *X* and *Y*. It is evident that this question is one of extending a given mapping. For if *f* and *g* are

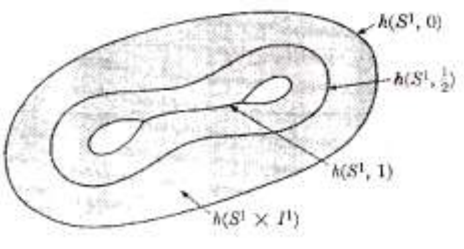

FIGURE 1.1

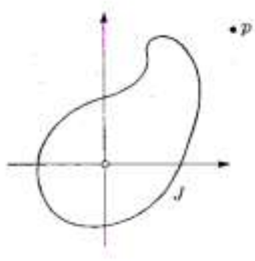

FIGURE 1.2

Two mappings of X into Y, then we have a mapping *h* on the closed subset $(X \times 0) \cup (X \times 1)$ of $X \times I^1$ given by $h'(x,0) = f(x)$ and $h'(x,1) = g(x)$.Then *f* and g are homotopic if and only if h can be extended to a mapping h of the entire product space $X \times I^1$ into Y. Thus it would seem that theorems about homotopy are but special cases of more general theorems on the extension of mappings. Indeed such is the case, but the general extension problem is far from being soled, and also the special case of homotopy plays an important role in the more general problem.

THEOREM1.1 The homotopy relation between mappings of a space X into a space Y is an equivalence relation on Y^X. That is, the relation "$\simeq$" satisfies:-

(1) $f \simeq f$ for each mapping f in Y^X (reflexive law),

(2) $f \simeq g$ implies $g \simeq f$ (symmetry law),

and

(3) $f \simeq g$ and $g \simeq k$ implies $f \simeq k$ (transitive law).

Proof: (1) For any mapping f in Y^X, define $h : W \times I^1 \to Y$ by

$$h(x,t) = f(x) \qquad (0 \le t \le 1).$$

It is evident that h is continuous and that $h(x,0) = f(x) = h(x,1)$ for all points x in X.

(2) If $f \simeq g$, then there is a homotopy $h : X \times I^1 \to Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all points x in X. We define

$$\bar{h}(x,t) = h(x,1-t).$$

Again $\bar{h}$ is obviously continuous and $\bar{h}(x,0) = g(x)$, while $\bar{h}(x,1) = f(x)$. Thus $g \simeq$f ,.

(3) If $f \simeq g$ and $g \simeq k$, then there are homotopies h_1 and h_2 with $h_1(x,0) = f(x), h_1(x,1) = g(x), h_2(x,0) = g(x),$ and $h_2(x,1) = k(x)$. We define a homotopy h between f and k by setting

$$\begin{aligned} h(x,t) &= h_1(x,2t) \quad \left(0 \le t \le \frac{1}{2}\right) \\ &= h_2(x,2t-1) \quad \left(\frac{1}{2} \le t \le 1\right). \end{aligned}$$

Then $h\left(x,\frac{1}{2}\right) = g(x)$ by both definitions, so h is well-defined and continuous on $X \times I^1$. Clearly $h(x,0) = h_1(x,0) = f(x),$ while $h(x,1) = h_2(x,1) = k(x)$. Thus $f \simeq k$.

THEOREM1.2. Let A be a closed subset of a separable metric M, and let $f^`$ and $g^`$ be homotopic mappings of A into the n-sphere S^n. It there exists an extension f of $f^`$ to all of M, then there also exists an extension g of g' to all of M, and the extensions f and g may be chosen to be homotopic also.

Proof (we follow Dowker [74]): Let $h: A \times I^1 \to S^n$ be the assumed homotopy between f' and g', and let f be the given extension of $f^`$ to all of M. Let D be the set in $M \times I^1$ given by

$$D = (A \times I^1) \cup (M \times 0).$$

Clearly D is a closed subset of $M \times I^1$, and on D we may define the mapping $F: D \to S^n$ given by

$$F(x,0) = f(x) \quad \text{for all } x \text{ in } M,$$

and

$$F(x,t) = h\ (x,t) \quad \text{for all } x \text{ in } A \text{ and } 0 \le t \le 1.$$

Since $h\ (x,0) = f(x) = f(x)$ for all points x in A, this mapping F is well-defined and continuous.

2. **The fundamental group**.

Let Y be a topological space, and let y_o be a point in Y. Then the y_o-neighborhood of curves in Y, C(Y,y_o), is the collection of all continuous mapping $f : I^1 \to Y$ of the unit interval into Y such that $f(0) = y_0 = f(1)$. Note that $C(Y, y_0)$ is a subspace of the function space Y^{I^1} and is not a neighborhood in Y in the usual sense.

Let f and g be two mapping in $C(Y, y_0)$. Then f is homotopic to g modulo y_o (abbreviated $f \underset{y_o}{\simeq} g$) if there exists a homotopy $h : I^1 \times I^1 \to Y$ such that

$$h(x,0) = f(x) \text{ and } h(x,1) = g(x) \quad \text{for all } x \text{ in } I^1$$

and $$h(0,t) = y_0 = h(1,t) \quad \text{for all } t \text{ in } I^1.$$

This is illustrated by Fig. 2.1

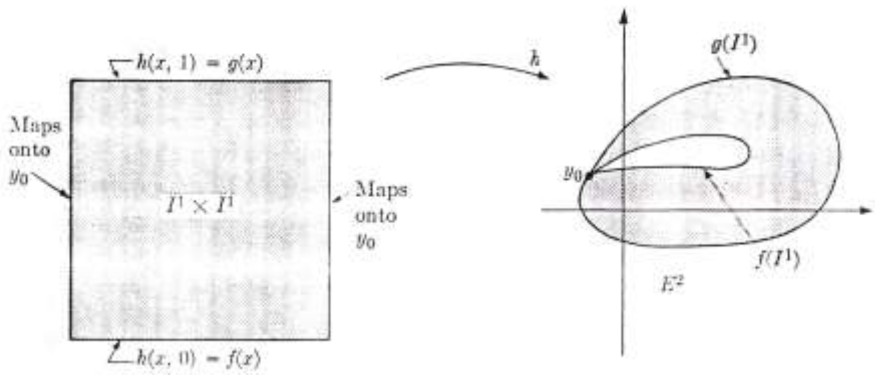

FIGURE 2.1

LEMMA. Homotopy modulo y_0 is an equivalence relation on $C(Y, y_0)$.

It has to be shown that homotopy modulo y_o is reflexive, symmetric, and transitive.

We let $\pi_1(Y, y_0)$ denote the collection of these equivalence classes. By introducing a suitable group operation, this collection becomes the fundamental group of Y modulo y_0 (or the Poincare group of Y or the first homotopy group of Y modulo y_0).

A homotopy$h$ between these two mapping may be given as follows:

$$h(x,t) = f_1\left(\frac{4x}{t+1}\right) \qquad \text{for pairs } (x,t) \text{ with } t \geq 4x-1$$

$$= f_2(4x-t-1) \qquad \text{for pairs } (x,t) \text{ with } 4x-1 \geq t \geq 4x-2$$

$$= f_3\left(\frac{4x-t-2}{2-t}\right) \qquad \text{for pairs } (x,t) \text{ with } 4x-2 \geq t.$$

It is a simple matter to check that h is the desired homotopy modulo y_0. For

$$\left.\begin{aligned} h(x,0) &= f_1(4x) && \text{for } 0 \geq 4x-1 \text{ or } 0 \leq x \leq \tfrac{1}{4} \\ &= f_2(4x-1) && \text{for } 4x-1 \geq 0 \geq 4x-2 \text{ or } \tfrac{1}{4} \leq x \leq \tfrac{1}{2} \\ &= f_3(2x-1) && \text{for } 4x-2 \geq 0 \text{ or } \tfrac{1}{2} \leq x \leq 1 \end{aligned}\right\} = (f_1 * f_2) * f_3$$

while

$$\left.\begin{aligned} h(x,1) &= f_1(2x) && \text{for } 1 \geqq 4x-1 \text{ or } 0 \leq x \leq \tfrac{1}{2} \\ &= f_2(4x-2) && \text{for } 4x-1 \geqq 1 \geqq 4x-2 \text{ or } \tfrac{1}{2} \leq x \leq \tfrac{3}{4} \\ &= f_3(4x-3) && \text{for } 4x-2 \geqq 1 \text{ or } \tfrac{3}{4} \leq x \leq 1 \end{aligned}\right\} = f_1 * (f_2 * f_3).$$

Since for $t = 4x-1,$ we have $h(x,t) = f_1(x),$ etc., the continuity of h is assured and the associative law has been proved.

Next, let j denote the constant mapping $j(x) = y_0$ for each point x in I^1. We claim that the equivalence class [j] is the identity element of $\pi_1(Y, y_0)$.To prove this, it will suffice to show that $f * j \underset{y_0}{\simeq} f$ for any function f in $C(Y, y_0)$.

This is done by constructing the homotpy

$$h(x,t)\, f\left(\frac{2x}{1+t}\right) \qquad \text{for pairs } (x,t) \text{ with } t \geqq 2r-1$$

$$= y_0 \qquad \text{for pairs (x,}t\text{) with } t \leqq 2x-1.$$

(To see where we got this, examine Fig. 2.3) The continuity of h is only in question where $t = 2x-1$, but for any such point, $h(x,t) = y_0$, so h is continuous as required. A check of the boundary conditions shows that

$$\left.\begin{aligned} h(x,0) &= f(2x) && \text{for } 0 > 2\text{x} - 1 \text{ or } 0 \leq \text{x} \leq \frac{1}{2} \\ &= y_0 && \text{for } 0 \leq 2\text{x} - 1 \text{ or } \frac{1}{2} \leq x \leq 1 \end{aligned}\right\} = f * j$$

and

$$h(x,1) = f(x) \qquad \text{for } 1 > 2x - 1 \text{ or } 0 \leq x \leq 1.$$

The other boundary conditions are obvious, and we know that [j] is the identity element of $\pi_1(Y, y_0)$.

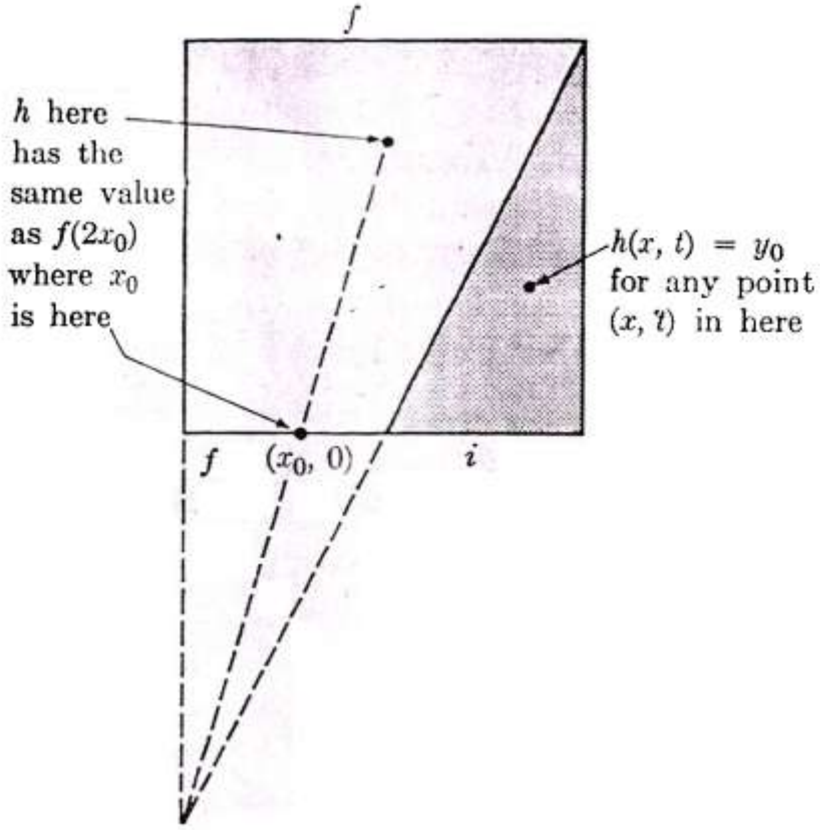

FIGURE 2.3

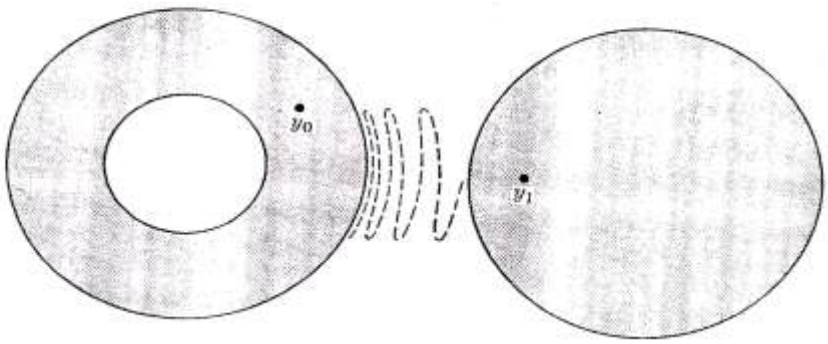

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FIGURE 2.4

and that for $t = 2x - 1$,

$$h(x,t) = f\left(\frac{2x-2}{2x-1-1}\right) = f(1) = y_0.$$

Thus*h* has the necessary continuity. The only question here concerns continuity at $t = 1$, but we need only insert the limiting values of the arguments to complete the argument. Checking the boundary conditions, we see that

$$\left.\begin{aligned} h(xo) &= f(2x) && \text{for } 0 \le 1\text{-}2x \text{ or } 0 \le x \le \frac{1}{2} \\ &= y_0 && \text{for } 2x\text{-}1 \le 0 \le 1\text{-}2x \text{ or } x = \frac{1}{2} \\ &= f\left(\frac{2x-2}{-1}\right) = f(2-2x) && \text{for } 0 \le 2x\text{-}1 \text{or } \frac{1}{2} \le x \le 1 \end{aligned}\right\} = f * \overline{f}$$

while

$h(x,1) = y_0$ for all *x* satisfying the inequalities,

We notice that the fundamental group as defined seems to depend upon the *base point*y_0 in *Y*, and in general this is true. If, for instance, Y is the union of an annular region in E^2 and a disjoint disc in E^2 (see Fig. 2.4), then for y_0 (any point in the annular region), $\pi_1(Y, y_0)$ is infinite cyclic, wheras if y_1 is any point in the disc, $\pi_1(Y, y_1)$ consists only of the identity element. One notes that this example fails to be connected and might conjecture that for a connected space, the group $\pi_1(Y, y_0)$ and $\pi_1(Y, y_1), y_0 \neq y_1$, would necessarily be isomorphic. It is easy to modity the above example by simply adding a sin (1/*x*) curve as the broken line in Fig. 2.4, and so disprove this conjecture.

3. Action on Groups

Suppose G and G are groups, written multiplicatively. A homomorphism $f : G \to G'$ is a map such that $f(x.y) = f(x).f(y)$ for all $x, y;$ it automatically satisfies the equations $f(e) = e'$ and $f(x^{-1}) = f(x)^{-1}$, where e end e' are the identities of G and G`, respectively, and the exponent-1 denotes the inverse. The kernel of *f* is the set $f^{-1}$ (e`); it is a subgroup of G. The image of f, similarly, is a subgroup of G`. The homeomorphism *f* is called a monomorphism if it is injective (or equivalently, if the kernel of f consists of e alone). It is called an epimorphism if it is subjective; and it is called an isomorphism if it is bijective.

Suppose G is a group and H is a subgroup of G. Let *x*H denote the set of all products *x*h, for h $\in$H; it is called a *left coset* of H in G. The collection of all such cosets forms a partition of G. Simiarly, the collection of all right cosets H *x* of H in G forms a partition of G. We call H a *normal subgroup* of G if $xhx^{-1} \in \mathrm{H}$ for each x $\in$ G and each h $\in$H. In this case, we have x.H = Hx for each x, so that our two partitions of G are the same.

Definition: Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a loop based at x_0. The set of pathhomotopy classes of loops based at x_0, with the operation, is called the *fundamental group* of X relative to the base point x_0. It is denoted $\pi_1(X, x_0)$

Theorem 3.1 The map $\hat{\alpha}$ is a group isomorphism.

Proof. To show that $\hat{\alpha}$ is a homomorphism, we compute

$$\hat{\alpha}[f]*\hat{\alpha}([g]) = ([\hat{\alpha}]*[f]*[\alpha])*[(\hat{\alpha}]*[g]*[\alpha])$$

$$=[\hat{\alpha}]*[f]*[g]*[\alpha]$$

$$=\hat{\alpha}([f]*[g]).$$

To show that $\hat{\alpha}$ is an isomorphism, we show that if β denotes that path $\hat{\alpha}$, which is the reverse of α, then $\hat{\beta}$ is an inverse for $\hat{\alpha}$. We compute, for each element [h] of $\pi_1(X,x_1)$.

$$\hat{\beta}([h]) - [\bar{\beta}]*[h]*[\beta] = [\alpha]*[h]*[\hat{\alpha}],$$

$$\hat{\alpha}\,\hat{\beta}([h]) = [\hat{\alpha}]*[\alpha]*[h] = [\hat{\alpha}]\Big)*[\alpha] = [h],$$

A similar computation shows that $\hat{\beta}(\hat{\alpha}([f]) = [f]$ for each $[f] \in \pi_1(X,x_0)$

Corollary If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X,x_0)$ is isomorphic to $\pi_1(X,x_1)$

Suppose that X is a topological space. Let C be the path component of X containing x_0. It is easy to see that $\pi_1(C,x_0) = \pi_1(X,x_0)$, since all loops and homotopies in X that are based at x_0 must lie in the subspace C. Thus $\pi_1(X,x_0)$ depends on only the path component of X containing x_0; it gives us no information whatever about the rest of X. For this reason, it is usual to deal with only path-connected spaces when studying the fundamental group.

It X is path connected, all the groups $\pi_1(X,x)$ are isomorphic, so it is temping to try to "identify" all those groups with one another and to speak simply of the fundamental group of the space X, without reference to base point. The difficulty with this approach is that there is no natural way of identifying $\pi_1(X,x_0)$ with $\pi_1(X,x_1)$; different paths α and β from x_0 to x_i may give rise to different isomorphism between these groups. For this reason, omotting the base point can lead to error.

It turns out that the isomorphism of $\pi_1(X,x_0)$ with $\pi_1(X,x_1)$ is idependent of path if and only if the fundamental group is abelian. This is a stringent requirement on the space X.

Definition. Let $h:(X,x_0) \to (Y,y_0)$ be a continuous map. Define

$$h:\pi_1(X,x_0) \to \pi_1(Y,y_0)$$

by the equation

$$h_*([f] = [h \text{ o } f].$$

The map h_* is called the *homomorphism induced by h*, relative to the base point x_0.

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Theorem 3.2 If $h:(X,x_0)\to(Y,y_0)$ and $k:(Y,y_0)\to(Z,z_0)$ are continuous, then $(K\ o\ h)* = k*\ ^0h*$If $i:(X,x_0)\to(X,x_0)$is the identity map, then I_* is the identity homomorphism

Proof The proof is a triviality. By definition

$$(k\ o\ h)_*([f]) = (k\ o\ h)\ o\ f],$$

$$(k_*\ o\ h_*)\ ([f])\ =\ K_*(h_*([f])) = k_*\ ([h\ o\ f])\ = [k\ o\ (h\ o\ f)]$$

Similarly, $i_*([f]) = [i\ o\ f] = [f]$

Theorem 3.3 Let $p:E\to B$be a covering map; let $p(e_0) := b_0$.

(a) The homomorphism $p_*:\pi_1(E,e_0)\to\pi_1(B,b_0)$ is a monomorphism

(b) Let $H = p.(\pi_1(E,e_0)$. The lifting correspondence ϕ induces an injective map.

$$\Phi\ :\ \pi_1(B,b_0)\big|H \to p^{-1}(b_0)$$

of the collection of right cosets of H into p^{-1} (b_0), which is bijective if E is path connected.

(c) If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to loop in E based at e_0.

Proof. (a) Suppose $\tilde{h}$ is a loop in E ate_0 and $p_*([\tilde{h}])$ is the identity element. Let F be a path homotopy between $p\ o\ \tilde{h}$ and the constant loop. If $\tilde{F}$ is the lifting of F to E such that $\tilde{F}\ (0,0) = e_0$, and $\tilde{F}$ is a path homotopy between $\tilde{h}$ and the constant loop at e_0.

(b) Given loop f and g in B, let $\tilde{f}$ and $\tilde{g}$ be liftings of them to E that begins at e_0.Then $\phi([f]) = \tilde{f}(1)$ and $\phi\ ([g]) = \tilde{g}(1)$. We show that $\phi([f]) = \phi\ ([g])$ if and only if $[f] \in H*[g]$.

First, suppose that $[f] \in H * [g]$. Then $[f] = [h * g]$, where $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ in E based at e_0. Now the product $\tilde{h} * \tilde{g}$ is defined, and it is a lifting of $h * g$ Because $[f] = [h * g]$, the lifting $\tilde{f}$ and $\tilde{h} * \tilde{g}$ which begin at e_0, must end at the same point of E. Then $\tilde{f}$ and $\tilde{g}$ end at he same point of E, so that $\phi([f]) = \phi([g])$.

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