

FUZZY β -IRRESOLUTE MAPPING

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ABSTRACT

Some properties of fuzzy β -irresolute mapping (formerly known as fuzzy $M\beta$ -continuous mapping [8]) have been studied here. Also it has been shown that fuzzy irresolute mapping [11] and fuzzy β -irresolute mapping are independent notions. In the last section some applications of fuzzy β -irresolute mapping have been discussed.

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INTRODUCTION

Throughout the paper, by (X, τ) or simply by X we mean a fuzzy topological space (fts, for short) in the sense of Chang [4]. A fuzzy set [16] A is a mapping from a nonempty set X into a closed interval $I = [0, 1]$. The support [13] of a fuzzy set A in X will be denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [13] with the singleton support $x \in X$ and the value t ($0 < t \leq 1$) at x will be denoted by x_t . 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 in X respectively. The complement [16] of a fuzzy set A in X will be denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for all $x \in X$. For two fuzzy sets A and B in X , we write $A \leq B$ if and only if $A(x) \leq B(x)$, for each $x \in X$, and AqB means A is quasi-coincident (q-coincident, for short) with B [13] if $A(x) + B(x) > 1$, for some $x \in X$. The negation of these two statements will be denoted by $A \not\leq B$ and $A\bar{q}B$ respectively. clA and $intA$ of a fuzzy set A in X respectively stand for the fuzzy closure [4] and fuzzy interior [4] of A in X . A fuzzy set A in X will be called fuzzy semiopen [2] (resp., fuzzy β -open [1], fuzzy preopen [12]) if $A \leq cl\ intA$ (resp., $A \leq$

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$cl\ int\ clA, A \leq int\ clA$). The set of all fuzzy semiopen (resp., fuzzy β -open) sets of X will be denoted by $SO(X)$ (resp., $\beta O(X)$). The complement of a fuzzy semiopen (resp., fuzzy β -open, fuzzy preopen) set A in X is called fuzzy semiclosed [2] (resp., fuzzy β -closed [1], fuzzy preclosed [12]). The smallest fuzzy semiclosed (resp., fuzzy β -closed, fuzzy preclosed) set containing a fuzzy set A in X is called fuzzy semiclosure [2] (resp., fuzzy β -closure [1], fuzzy preclosure [12]) of A and is denoted by $sclA$ (resp., $\beta clA, pclA$). A fuzzy set B in X is said to be a β -nbd [1] of a fuzzy point x_t in X if there exists a fuzzy β -open set U in X such that $x_t \leq U \leq B$. A fuzzy set B is called a fuzzy β -q-nbd [8] of a fuzzy point x_t in X if there is a fuzzy β -open set U in X such that $x_t qU \leq B$.

1. FUZZY β -IRRESOLUTE MAPPING : SOME CHARACTERIZATIONS

In this section fuzzy β -irresolute mapping has been characterized in different ways.

DEFINITION 1.1. A fuzzy mapping $f : X \rightarrow Y$ is said to be fuzzy β -irresolute (fuzzy $M\beta$ -continuous mapping [8]) if $f^{-1}(A)$ is fuzzy β -open in X for each fuzzy β -open set A in Y .

THEOREM 1.2. Let $f : X \rightarrow Y$ be a fuzzy function. Then the following are equivalent :

- (a) f is fuzzy β -irresolute,
- (b) for each fuzzy point x_t in X and each fuzzy β -open set A in Y such that $f(x_t) \leq A$, there exists a fuzzy β -open set B in X such that $x_t \leq B$ and $f(B) \leq A$,
- (c) $f^{-1}(B)$ is fuzzy β -closed in X for each fuzzy β -closed set B in Y ,
- (d) for each fuzzy point x_t in X , the inverse of each fuzzy β -nbd B of $f(x_t)$ in Y is a fuzzy β -nbd of x_t in X ,
- (e) for each fuzzy point x_t in X and each fuzzy β -nbd B of $f(x_t)$, there exists a fuzzy β -nbd C of x_t in X such that $f(C) \leq B$,
- (f) for each fuzzy set D in X , $f(\beta cl D) \leq \beta cl f(D)$,
- (g) for each fuzzy set B in Y , $\beta cl (f^{-1}(B)) \leq f^{-1}(\beta cl B)$.

PROOF. (b) \Rightarrow (a). Let A be a fuzzy β -open set in Y and x_t , fuzzy point in $f^{-1}(A)$. Then $x_t \leq f^{-1}(A)$, i.e., $f(x_t) \leq A$. By (b), there exists a fuzzy β -open set B in X such that $x_t \leq B$ and $f(B) \leq A$. Thus $B \leq f^{-1}(A)$. We have to show that $f^{-1}(A) \leq cl\ int\ cl f^{-1}(A)$. As $B \in \beta O(X)$, $x_t \leq B \leq cl\ int\ cl B \leq cl\ int\ cl f^{-1}(A)$. As $x_t \leq f^{-1}(A)$, $f^{-1}(A) \leq cl\ int\ cl f^{-1}(A)$.

(a) \Rightarrow (c). Let B be any fuzzy β -closed set in Y . Then $1_Y \setminus B \in \beta O(Y)$. By (a), $f^{-1}(1_Y \setminus B) = 1_X \setminus f^{-1}B \in \beta O(X)$ and so $f^{-1}(B)$ is fuzzy β -closed in X .

(c) \Rightarrow (a). Straightforward.

(a) \Rightarrow (d). Let x_t be a fuzzy point in X and B , a fuzzy β -nbd of $f(x_t)$ in Y . Then there exists $U \in \beta O(Y)$ such that $f(x_t) \leq U \leq B$. Then $x_t \leq f^{-1}(U) \leq f^{-1}(B)$. Since $U \in \beta O(Y)$, by (a) $f^{-1}(U) \in \beta O(X)$ and hence the result.

(d) \Rightarrow (e). Since $f f^{-1}(B) \leq B$, the result follows by taking $C = f^{-1}(B)$.

(e) \Rightarrow (b). Let x_t be a fuzzy point in X and A , any fuzzy β -open set in Y such that $f(x_t) \leq A$. Then A is fuzzy β -nbd of $f(x_t)$ in Y . By (e), there exists a fuzzy β -nbd C of x_t in X such that $f(C) \leq A$. Therefore, there exists $U \in \beta O(X)$ such that $x_t \leq U \leq C$ and so $f(U) \leq f(C) \leq A \Rightarrow f(U) \leq A$.

(c) \Rightarrow (f). Let D be any fuzzy set in X . Then $\beta cl f(D)$ is fuzzy β -closed in Y . By (c), $f^{-1}(\beta cl f(D))$ is fuzzy β -closed in X . Now $D \leq f^{-1}f(D) \leq f^{-1}(\beta cl f(D))$, i.e., $\beta cl D \leq \beta cl f^{-1}(\beta cl f(D)) = f^{-1}(\beta cl f(D))$. Therefore, $f(\beta cl D) \leq \beta cl f(D)$.

(f) \Rightarrow (c). Let B be any fuzzy β -closed set in Y . Put $D = f^{-1}(B)$. By (f), $f(\beta cl D) \leq \beta cl f(D) = \beta cl (f(f^{-1}(B))) \leq \beta cl B = B$. Thus $\beta cl D \leq f^{-1}(f(\beta cl D)) \leq f^{-1}(B) = D$. Hence $D = f^{-1}(B)$ is fuzzy β -closed in X .

(f) \Rightarrow (g). Let $B \in I^Y$. Again let $D = f^{-1}(B)$. By (f), $f(\beta cl D) \leq \beta cl f(D)$, i.e., $\beta cl D \leq f^{-1}(\beta cl f(D))$, i.e., $\beta cl f^{-1}(B) \leq f^{-1}(\beta cl f(f^{-1}(B))) \leq f^{-1}(\beta cl B)$.

(g) \Rightarrow (f). Let $D \in I^X$. By (g), $\beta cl (f^{-1}f(D)) \leq f^{-1}(\beta cl f(D)) \Rightarrow \beta cl D \leq f^{-1}(\beta cl f(D)) \Rightarrow f(\beta cl D) \leq \beta cl f(D)$.

THEOREM 1.3. A mapping $f : X \rightarrow Y$ is fuzzy β -irresolute iff for each fuzzy point x_t in X and any fuzzy β -open β -q-nbd V of $f(x_t)$ in Y , there exists a fuzzy β -open β -q-nbd U of x_t in X such that $f(U) \leq V$.

PROOF. Let $f : X \rightarrow Y$ be fuzzy β -irresolute and x_t be a fuzzy point in X . Let V be a fuzzy β -open β -q-nbd of $f(x_t)$ in Y . Then $f^{-1}(V)$ ($= U$, say) is a fuzzy β -open β -q-nbd of $f(x_t)$ in X such that $f(U) \leq V$.

Conversely, let x_t be any fuzzy point in X and V be any fuzzy β -open set containing $f(x_t)$. Let m_t be a positive integer such that $1/m_t < t$. Then $0 < 1 - t + 1/n = \beta_n$ (say) < 1 , for all $n \geq m_t$. Now $y_{\beta_n} qV$ for each $n \geq m_t$, where $y = f(x)$. Then by hypothesis, there exists a fuzzy β -open set U_n in X such that $x_{\beta_n} qU_n$ and $f(U_n) \leq V$, for all $n \geq m_t$. Put $U = \bigcup_{n \geq m_t} U_n$. Then $U \in \beta O(X)$ such that $f(U) \leq V$. Also $\beta_n + U_n(x) > 1$, for all $n \geq m_t \Rightarrow 1 - t + 1/n + U_n(x) > 1$, for all $n \geq m_t \Rightarrow t < U_n(x) + 1/n$, for all $n \geq m_t \Rightarrow t \leq \sup\{U_n(x) : n \geq m_t\} = U(x) \Rightarrow x_t \leq U$. Hence by Theorem 1.2, f is fuzzy β -irresolute.

2. FUZZY IRRESOLUTE AND FUZZY β -IRRESOLUTE MAPPING

In this section it has been shown that fuzzy irresolute mapping [11] and fuzzy β -irresolute mapping are independent notions.

First we recall the definition from [11] for ready reference.

DEFINITION 2.1. A fuzzy mapping $f : X \rightarrow Y$ is said to be fuzzy irresolute if $f^{-1}(A)$ is fuzzy semiopen in X for each fuzzy semiopen set A in Y .

REMARK 2.2. It is clear from the following two examples that fuzzy irresolute mapping and fuzzy β -irresolute mapping are independent notions.

EXAMPLE 2.3. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, C\}$ where $A(a) = 0.5$, $A(b) = 0.4$, $C(a) = 0.6$, $C(b) = 0.5$. Then (X, τ) and (X, τ_1) are fts's. Consider the fuzzy mapping $f : (X, \tau) \rightarrow (X, \tau_1)$ defined by $f(a) = b$, $f(b) = a$. We claim that f is fuzzy β -irresolute but not fuzzy irresolute mapping. The collection of all fuzzy semiopen sets in (X, τ) is $\{0_X, 1_X, A, U\}$ where $A \leq U \leq 1_X \setminus A$ and that of in (X, τ_1) is $\{0_X, 1_X, C, V\}$ where $V \geq C$. Again any fuzzy set in (X, τ) is fuzzy β -open in (X, τ) and the collection of all fuzzy β -open sets in (X, τ_1) is $\{0_X, 1_X, C, W\}$ where $W \not\leq 1_X \setminus C$.

Let B be a fuzzy semiopen set in (X, τ_1) defined by $B(a) = B(b) = 0.6$. Now $[f^{-1}(B)](a) = B f(a) = B(b) = 0.6$, $[f^{-1}(B)](b) = B f(b) = B(a) = 0.6$, and so $f^{-1}(B) \notin SO(X, \tau)$. Therefore, f is not fuzzy irresolute mapping. Since any fuzzy set in (X, τ) is fuzzy β -open in (X, τ) , f is fuzzy β -irresolute.

EXAMPLE 2.4. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, B\}$ where $A(a) = 0.4, A(b) = 0.7, B(a) = 0.6, B(b) = 0.7$. Then (X, τ) and (X, τ_1) are fts's. Now fuzzy semiopen sets in (X, τ) are $0_X, 1_X, A, V$ where $V \geq A$ and that of fuzzy β -open sets in (X, τ) are $0_X, 1_X, A, U$ where $U \not\leq 1_X \setminus A$. Again fuzzy semiopen sets in (X, τ_1) are $0_X, 1_X, B, C$ where $C \geq B$ and that of fuzzy β -open sets in (X, τ_1) are $0_X, 1_X, B, W$ where $W \not\leq 1_X \setminus B$. Consider the fuzzy identity mapping $i : (X, \tau) \rightarrow (X, \tau_1)$. We claim that i is fuzzy irresolute but not fuzzy β -irresolute mapping. Infact, $[i^{-1}(C)](a) = C(i(a)) = C(a) \geq B(a)$ and $[i^{-1}(C)](b) = C(i(b)) = C(b) \geq B(b)$ and $B \geq A \Rightarrow i^{-1}(C) \geq A$ which shows that i is fuzzy irresolute. But $W(a) = 0.6, W(b) = 0.3$ being a fuzzy β -open set in (X, τ_1) and $i^{-1}(W) = W \notin \beta O(X, \tau)$ and so i is not fuzzy β -irresolute mapping.

3. APPLICATIONS

Let us recall some definitions for ready references.

DEFINITION 3.1 [4]. Let A be a fuzzy set in an fts X . A collection \mathcal{U} of fuzzy sets in X is called a fuzzy cover of A if $\sup\{U(x) : U \in \mathcal{U}\} = 1$, for each $x \in \text{supp}A$. In particular, if $A = 1_X$, we get the definition of fuzzy cover of the fts X .

DEFINITION 3.2 [6]. A fuzzy cover \mathcal{U} of a fuzzy set A in an fts X is said to have a finite subcover \mathcal{U}_0 if \mathcal{U}_0 is a finite subcollection of \mathcal{U} such that $\cup \mathcal{U}_0 \geq A$. In particular, if $A = 1_X$, then the requirement on \mathcal{U}_0 is $\cup \mathcal{U}_0 = 1_X$.

DEFINITION 3.3 [9]. An fts X is said to be a fuzzy semicompact space if every cover of X by fuzzy semiopen sets has a finite subcover.

DEFINITION 3.4. An fts X is said to be fuzzy S -closed [10] (resp., fuzzy s -closed [15]) if every fuzzy cover \mathcal{U} of X by fuzzy semiopen sets in X has a finite subfamily \mathcal{U}_0 such that $\cup_{U \in \mathcal{U}_0} clU = 1_X$ (resp., $\cup_{U \in \mathcal{U}_0} sclU = 1_X$).

DEFINITION 3.5 [3]. An fts X is said to be fuzzy β -compact space if every fuzzy cover of X by fuzzy β -open sets in X has a finite subcover.

DEFINITION 3.6 [8]. An fts X is said to be fuzzy β -closed if for every fuzzy cover \mathcal{U} of X by fuzzy β -open sets in X , there exists a finite subfamily \mathcal{U}_0 of \mathcal{U} such that $\bigcup_{U \in \mathcal{U}_0} \beta cl U = 1_X$.

DEFINITION 3.7. An fts X is said to be fuzzy strongly compact [12] (resp., fuzzy P -closed [17]) if every cover of X by fuzzy preopen sets in X has a finite subcover (resp., subfamily \mathcal{U}_0 of \mathcal{U} such that $\bigcup_{U \in \mathcal{U}_0} pcl U = 1_X$).

THEOREM 3.8. If X is a fuzzy β -compact space and $f : X \rightarrow Y$ is fuzzy β -irresolute surjective mapping, then Y is fuzzy semicompact.

PROOF. Let $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of Y by fuzzy semiopen sets of Y . Then as fuzzy semiopen sets are fuzzy β -open, \mathcal{V} is a fuzzy cover of X by fuzzy β -open sets of Y . Now f being fuzzy β -irresolute surjective mapping, $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is a fuzzy cover of X by fuzzy β -open sets of X . As X is fuzzy β -compact, there exists a finite subfamily Λ_0 of Λ such that $\{f^{-1}(V_\alpha) : \alpha \in \Lambda_0\}$ also covers X , i.e., $1_X = \bigcup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha) \Rightarrow 1_Y = f(1_X) = f(\bigcup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha)) = f f^{-1}(\bigcup_{\alpha \in \Lambda_0} V_\alpha) \leq \bigcup_{\alpha \in \Lambda_0} V_\alpha$. Hence Y is fuzzy semicompact space.

REMARK 3.9. Since fuzzy semicompact space is fuzzy S -closed space, we can state the following theorem.

THEOREM 3.10. If X is fuzzy β -compact space and $f : X \rightarrow Y$ is fuzzy β -irresolute surjective mapping, then Y is fuzzy S -closed space.

PROOF. The proof is same as that of Theorem 3.8 and hence omitted.

REMARK 3.11. Since fuzzy preopen set is fuzzy β -open, we can state the following theorem.

THEOREM 3.12. If X is fuzzy β -compact space and $f : X \rightarrow Y$ is fuzzy β -irresolute surjective mapping, then Y is fuzzy strongly compact (resp., fuzzy P -closed).

REMARK 3.13. Since for a fuzzy set A in X , $\beta cl A \leq scl A$, $\beta cl A \leq pcl A$, $\beta cl A \leq cl A$, we can easily state the following theorem.

THEOREM 3.15. If X is fuzzy β -closed space and $f : X \rightarrow Y$ is fuzzy β -irresolute surjective mapping, then Y is fuzzy S -closed (resp., fuzzy s -closed, fuzzy P -closed) space.

NOTE 3.16. Instead of space we can state the Theorem 3.8, Theorem 3.10, Theorem 3.12, Theorem 3.14 for a fuzzy set $A \in I^X$ also.

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