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CONCEPT OF AN ARBITRARY POWERS OF DOUBLE BAND MATRICES

Kunwar Pal Singh, Associate Professor

Department of Mathematics CCR (PG) College, Muzaffarnagar- 251001

Abstract

One of the most effective and powerful tools for the study of smmability and matrix theory is the development of several difference operators, their related sequence spaces and their various applications. The applications of difference operators become more apparent in study of several summable matrices and related properties mostly involving inversion, powers and norm of matrices in Linear algebra and the study of derivatives of arbitrary orders and their dynamic natures in Fractional calculus. These operators are also being used in study of spectral properties of different matrices and related eigenvalue problems in operator theory and many others. In this article we studied an arbitrary powers of double band matrices.

Keywords: *Summability, matrices, space etc.*

1. INTRODUCTION

In this paper, we define two new fractional difference operators and determine the explicit formulas for any arbitrary power of double band matrices. Subsequently, adaptive algorithms for finding the arbitrary powers of both upper and lower double band matrices have been developed [1]. Respective programming codes for the new algorithms have been constructed and verified by implementing them in MATLAB. Some numerical examples are also given in support to the new proposed programming codes [2].

Let $A = (a_{ij})$ $(i, j \in N)$ be a non-singular matrix of order n, $(n \in N)$ i.e.,

$$A := \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

Then, matrix A can be factorized as the product of a unit lower triangular matrix L and an upper triangular matrix U , i.e., A = LU, [3].

Suppose the matrices L(a, b) = (l_{ij}) and U(a, b) = (u_{ij}) denote the lower and upper double band matrices, respectively, then for $a \neq 0$, we write

$$l_{ij} = \begin{cases} a, & (j = i) \\ b, & (j = i - 1) \\ 0 & \text{otherwise} \end{cases} \text{ and } u_{ij} = \begin{cases} a, & (j = i) \\ b, & (i = j - 1) \\ 0 & \text{otherwise.} \end{cases}$$

The idea of difference operators of order one was initially provided by Kızmaz [7] and Altay and Basar [1] and further these were extended to the case of positive integer m. On generalization of the difference matrix Δ, it is defined the double band matrix B(r, s), where $r \neq 0$, $s \neq 0 \in R$ and studied the related sequence spaces and their spectral properties. Recently, for a proper fraction α , it is defined fractional difference operator $\Delta\alpha$, which not only generalizes the most of difference operators defined earlier [4], but it also provides some new and interesting ideas regarding fractional power of certain matrices, fractional derivatives of some functions and many others. Motivated by the earlier works, the main objective of this note is to define a fractional difference operator's analog to the double band matrix B(r, s) and establish certain results on finding powers of these matrices [5].

Let $x = (x_k)$ be any sequence in w and $a(\neq 0)$, b be two real numbers, then we define the generalized difference operators which generate matrices L(a, b) and U(a, b) as [6]

$(L(a, b)x)_k = ax_k + bx_{k+1}$; $(k \in N)$

And

$(U(a, b)x)_k = ax_k + bx_{k-1}$, ; $(k \in N)$.

Particularly, the matrices L(a, b) and U(a, b) generalize the difference operators of order one Δ and $\Delta^{(1)}$, respectively under the case a = 1 and b = −1 , where

$$\Delta = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \Delta^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is also observed that the m th power of matrices Δ and $\Delta^{(1)}$ are being calculated by taking difference operators Δ^m and $\Delta^{(m)}$ for all $m \in N$, respectively. One of the interesting calculations involving arbitrary power ($\alpha \in R$) of these matrices are also being calculated by taking difference operators Δ^α and $\Delta^{(\alpha)}$. Now, it is trivial to check that [8]

$$(\Delta)^\alpha = \begin{pmatrix} 1 & -\alpha & \frac{\alpha(\alpha-1)}{2!} & -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} & \dots \\ 0 & 1 & -\alpha & \frac{\alpha(\alpha-1)}{2!} & \dots \\ 0 & 0 & 1 & -\alpha & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } (\Delta^{(1)})^\alpha = ((\Delta)^\alpha)^T,$$

where A^T represents the transpose of A.

On generalizing all the difference operators discussed above, we define

$$(U^\alpha(a,b)x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)} a^{\alpha-i} b^i x_{k+i}, \ (k \in \mathbb{N}) \tag{1}$$

$$(L^{\alpha}(a,b)x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)} a^{\alpha-i} b^i x_{k-i}, \ (k \in \mathbb{N}). \quad (2)$$

where $\Gamma(\alpha)$ denotes the well known Gamma function of a real number $\alpha > 0$. For any integral values of α , Eqns.(1) and (2) reduce to the finite sums. Now, we state some numerical examples on certain sequences via these operators, [8].

Recently, Baliarsingh et al., [11] studied the fractional powers of double band matrices using fractional difference operators. Also, in that paper certain theoretical results on finding arbitrary powers of a matrix have been discussed, [9]. The main objective of this paper is to give a technical and numerical treatment to these results.

We state following theorems involving the integral and non-integral powers of double band matrices U(r, s) and L(r, s), [10].

Lemma 1. Let the lower double band matrix L = (l_{nk}) be defined by

$$l_{nk} = \begin{cases} r, & (k=n) \\ s, & (k=n-1) \\ 0, & (otherwise). \end{cases}$$

then for $\alpha \in R$, L^{α} = (l^{α}_{nk}) is given by

$$l^{\alpha}_{nk} = \begin{cases} r^{\alpha}, & (k=n) \\ \frac{\Gamma(\alpha+1)}{(n-k)!\Gamma(\alpha-n+k+1)} r^{\alpha-n+k} s^{n-k}, & (0 \le k < n) \\ 0, & (k > n). \end{cases}$$

Lemma 2. Let the upper double band matrix U = (u_{nk}) be defined by

$$u_{nk} = \begin{cases} r, & (k = n) \\ s, & (k = n+1) \\ 0, & (otherwise). \end{cases}$$

then for α ∈ R , $U^{\alpha} = u_{nk}^{\alpha}$ is given by

$$l_{nk}^{\alpha} = \begin{cases} 1, & (k = n) \\ \frac{\Gamma(\alpha+1)}{(n-k)!\Gamma(\alpha-n+k+1)} c^{n-k}, & (0 \leq k < n) \\ 0, & (k > n). \end{cases}$$

2. MAIN RESULTS

Now here, we provide some applications of Lemmas 1 and 2 .

Theorem 2.1. The arbitrary power α of the lower double band matrix L(1, c) is given by $L^{\alpha}(1, c) = l_{nk}^{\alpha}$, where

$$l_{nk}^{\alpha} = \begin{cases} 1, & (k = n) \\ \frac{\Gamma(\alpha+1)}{(n-k)!\Gamma(\alpha-n+k+1)} c^{n-k}, & (0 \leq k < n) \\ 0, & (k > n). \end{cases}$$

Proof. The proof is a direct consequence of Lemma 1.

Theorem 2.2. The arbitrary power α of the lower double band matrix U(a, b) is given by U^{α} (a, b) = u_{nk}^{α} where [12]

$$u_{nk}^{\alpha} = \begin{cases} a^{\alpha}, & (k = n) \\ \frac{\Gamma(\alpha+1)}{(k-n)!\Gamma(\alpha-k+n+1)} a^{\alpha-k+n} b^{k-n}, & (k > n) \\ 0, & (0 \leq k < n). \end{cases}$$

Proof. The proof follows from Lemma 2.

Theorem 2.3. Let A be a nonsingular matrix of order n and A = LU , then for any real $\alpha \neq 0, -1$, the following equation holds

$$A^{\alpha} \neq U^{\alpha} L^{\alpha}. \quad (3)$$

However, if A = L(1, c)U(a, b), then the explicit formula for $A^{-1} = \alpha_{ij}^{-1}$ is given by

$$a_{ij}^{-1} = \begin{cases} (-1)^{i+j} a^{-1-j+i} b^{j-i} + \sum_{k=j+1}^{n} (-1)^{i+j} a^{-1-k+i} b^{k-i} c^{k-j} & (i \leq j) \\ (-1)^{i+j} c^{i-j} a^{-1} + \sum_{k=i+1}^{n} (-1)^{(i+j)} a^{-1-k+i} b^{k-i} c^{k-j}. & (i > j) \end{cases}$$

Proof. The entire proof is divided into two parts. For better clarification of the first statement Eqn (3) we provide the following counter example: Consider a non singular matrix A of order 5, where

$$A = \begin{pmatrix} 9 & 2 & 0 & 0 & 0 \\ 45 & 19 & 2 & 0 & 0 \\ 0 & 45 & 19 & 2 & 0 \\ 0 & 0 & 45 & 19 & 2 \\ 0 & 0 & 0 & 45 & 19 \end{pmatrix}$$

On LU factorization of the tridiagonal type matrix A , we have A = LU, where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} 9 & 2 & 0 & 0 & 0 \\ 0 & 9 & 2 & 0 & 0 \\ 0 & 0 & 9 & 2 & 0 \\ 0 & 0 & 0 & 9 & 2 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix}$$

Clearly, using Theorems 2.1 and 2.2, square roots of the matrix L and U are being calculated directly and

$$L^{1/2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5/2 & 1 & 0 & 0 & 0 \\ -25/8 & 5/2 & 1 & 0 & 0 \\ 125/16 & -25/8 & 5/2 & 1 & 0 \\ -3225/128 & 125/16 & -25/8 & 5/2 & 1 \end{pmatrix}$$

And

$$U^{1/2} = \begin{pmatrix} 3 & 1/3 & -1/54 & 1/486 & -5/17496 \\ 0 & 3 & 1/3 & -1/54 & 1/486 \\ 0 & 0 & 3 & 1/3 & -1/54 \\ 0 & 0 & 0 & 3 & 1/3 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

It can be easily shown that

$A^{1/2} \neq U^{1/2}L^{1/2}$.

However, the equality holds in Equation (3) for $\alpha = -1$. In fact, for any natural number α the equality sign holds provided the matrices L and U commute.

3. CONCLUSION

Fractional difference operators analog to double band matrices have been introduced using which new algorithms for integral and non-integral powers of double band matrices have been proposed. The corresponding MATLAB programming has been constructed. As one of its applications, it is being used for finding the inverse of tri-diagonal type matrices. Furthermore, using the proposed difference operators, one may also define related sequence spaces and study their topological and geometrical properties. The spectral properties of these operators are yet to be studied which may generalize the notion of fine spectra of all double band matrices and difference operators of any arbitrary orders.

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