International Research Journal of Mathematics, Engineering and IT
ISSN: (2349-0322)
Impact Factor- 5.489 Volume 7, Issue 12, December 2020
Website- www.aarf.asia, **Email** : editor@aarf.asia , editoraarf@gmail.com

ADVANCES IN ENVELOPING SUMMABILITY THEORY OF SOME SEQUENCE SPACES

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Abstract

The theory of sequence spaces depends upon various result from topological vector spaces and their ramifications. In several branches of analysis, the study of sequence spaces occupies a very vigorous position. One can sees the construction of numerous examples of locally convex spaces obtained as a consequence of the dual structure displayed by several pairs of distinct sequence spaces. We rediscover the wide applicability of sequence spaces to several branches of functional analysis. In this article, a brief survey of sequence spaces, subsititution operators and an outline of the thesis are given. The influence of this study can be appreciated and applied in many aspects of Mathematics. The theory of sequence spaces has made remarkable advances in enveloping summability theory via unified techniques effecting matrix transformations from one sequence space into another.

Keywords: *Summability, Sequence, Space etc.*

1.INTRODUCTION

In functional analysis and related areas of Mathematics, a sequence space is a vector space whose elements are sequences of real or complex numbers. Equivalently, it is a function space whose elements are functions from the set of natural numbers to the field K of real or complex numbers. Apart from this, it is a powerful tool for obtaining positive results concerning Schauder bases and their associated types [3].

The extension of single sequence spaces to double sequence spaces is described as follows.

2. DOUBLE SEQUENCE SPACES

The initial work on double sequence spaces due to Bromwich. Later on, it was studied by Moricz [4], Tripathy [2], Basarir and Sonalcan [6] and many others. Quite recently, Zeltser[9] studied both the theory of topological and theory of summability of double sequences. Mursaleen and Edely [7] have recently introduced the statistical convergence and Cauchy convergence for double sequences and gave the relation between statistical convergent and strongly Ces`arosummable double sequences. Nextly, Mursaleen [8] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem. Later on, it is introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{mn})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [1] have defined the spaces BS, BS(t), CSp, CSbp, CSr and BV of double sequences consisting of all double series whose sequence of partial sums are in the spaces Mu, Mu(t), Cp, Cbp, Cr and Lu, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces BS, BV, CSbp and the β(v)-duals of the spaces CSbp and CSr of double series. Now, recently it has introduced the Banach space Lq of double sequences corresponding to the well known space `q of single sequences and examined some properties of the space Lq [5]. By the convergence of a double sequence we mean the convergence of the Pringsheim sense i.e. a double sequence $x = (x_{kl})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in N$ such that $|x_{kl} - L| < \varepsilon$ whenever $k, l > n$. We shall write more briefly as P-convergent. The double sequence $x = (x_{kl})$ is bounded if there exists a positive number M such that $|x_{kl}| < M$ for all k and l [10].

3. DOUBLE CHI SEQUENCE SPACES

By the double chi sequence we consider the sequences in Pringsheim sense means that a double sequence $x = (x_{mn}) \in E$ has Pringsheim limit 0 (denoted by $P - \lim x = 0$) if $((m+n)!|x_{mn})|)^{1/m+n} \to 0$, whenever $m, n \to \infty$. We shall denote the space of all P-chi sequence spaces by χ^2. The double sequence x is analytic if there exists a positive number M such that $|x_{mn}|^{1/ m+n} < M$ for all m, n. We will denote the set of all analytic double sequence spaces by Λ^2[11].

4. FUZZY SEQUENCE SPACES

In the classical set theory, it is not permissible for an individual to be partially in a set and also not in the same set at the same time. Thus many real-world application problems cannot be described and as a result, cannot be solved by the classical set theory, including all those involving elements with only partially membership of a set. On the contrary, fuzzy set theory accepts partially membership, therefore in a sense it generalizes the classical set theory to some extent. Fuzzy set theory offers as a new angle to observe and investigate the relation between sets and their elements other than traditional way [12].

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set operations were first introduced and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming.

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $X : R^n \to [0, 1]$ which satisfies the following four conditions:

(i) X is normal, i.e., there exist an $x0 \in R^n$ such that $X(x0) = 1$,

(ii) X is fuzzy convex, i.e., for $x, y \in R^n$ and $0 \le \lambda \le 1$, $X(\lambda x + (1 - \lambda)y) \ge X(x) \wedge Y(y) = \min[X(x), X(y)]$,

(iii) X is upper semi-continuous; i.e., if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$ for all $a \in [0, 1]$ is open in the usual topology of R^n ,

(iv) The closure of $\{x \in R^n: X(x) > 0\}$, denoted by $[X]^0$, is compact.

Let $C(R^n) = \{A \subset R^n: A \text{ is compact and convex }\}$. The spaces $C(R^n)$ has a linear structure induced by the operations

$A + B = \{a + b, a \in A, b \in B\}$

And

$\lambda A = \{\lambda a : a \in A\}$

for A, B ∈ C(R^n) and λ ∈ R. The Hausdorff distance between A and B of C(R^n) is defined as

$$\delta_\infty(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \ \sup_{b\in B} \inf_{a\in A} \|a - b\|\}$$

where $||.||$ denotes the usual Euclidean norm in R^n . It is well known that (C(R^n), δ_∞) is a complete (non separable) metric space.

For $0 < \alpha \leq 1$, the α-level set, $X^\alpha = \{x \in R^n: X(x) \geq \alpha\}$ is a non empty compact convex, subset of R^n , as is the support X^0. Let L(R^n) denote the set of all fuzzy numbers. The linear structure of L(R^n) induces addition X + Y and scalar multiplication λX, λ ∈ R, in terms of α-level sets, by

$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$

And

$[\lambda X]^\alpha = \lambda[X]^\alpha$

for each $0 \leq \alpha \leq 1$. Define for each $1 \leq q < \infty$

$$d_q(X, Y) = \left\{ \int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha \right\}^{1/q}$$

and $d_\infty(X, Y) = \sup \delta_\infty(X^\alpha, Y^\alpha)$ $\{0 \leq \alpha \leq 1\}$. Clearly $d_\infty(X, Y) = \lim_{q\to\infty} d_q(X, Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover (L(R^n), d_∞) is a complete, separable and locally compact metric space.

In Banach space theory, the study of Orlicz and Lorentz sequence spaces was initiated with certain specific purpose. It was the one who got keen involvement in studingOrlicz spaces in connection with finding Banach spaces with symmetric shauder basis having complimentary subspaces isomorphic to c0 or $l_p(1 \leq p < \infty)$. Orlicz sequence spaces was studied in details and solved many important and interesting structural problems in Banach spaces. Later on it is generalized the concept of Orlicz sequence spaces to modular sequence spaces which further help him to deeply understand the result of Lindberg and of Lindenstrauss and Tzafriri. It is shown Orlicz sequence space as a special case of Orlicz spaces and extensively studied. Orlicz spaces find a number of useful application in the theory of nonlinear integral equations whereasOrlicz sequence spaces are generalization of L p−spaces, L p−spaces find themselves enveloped in Orlicz spaces [13].

5. ORLICZ FUNCTION

An Orlicz function M : [0,∞) → [0,∞) is convex and continuous such that M(0) = 0, M(x) > 0 for x > 0.

Let w be the space of all real or complex sequences x = (x_k). it used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

is known as an Orlicz sequence space. The space l_M is a Banach space with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

It was shown that every Orlicz sequence space `M contains a subspace isomorphic to `p(p ≥ 1). In the later stage different Orlicz sequence spaces were introduced and studied. An Orlicz function M satisfies Δ2−condition if and only if for any constant L > 1 there exists a constant K(L) such that M(Lu) ≤ K(L)M(u) for all values of u ≥ 0. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is a right differentiable for t ≥ 0, η(0) = 0, η(t) > 0, η is non-decreasing and η(t) → ∞ as t → ∞.

Musielak-Orlicz Function

A sequence $M = (M_k)$ of Orlicz function is called a Musielak-Orlicz function. A sequence $N = (N_k)$ is defined by

$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, k = 1, 2, \cdots$ is called the complementary function of a Musielak-Orlicz function M

is called the complementary function of a Musielak-Orlicz function M. The sequence space t_M and its subspace h_M for a given Musielak-Orlicz function M, are defined as follows

$t_M = \{x \in w : I_M(cx) < \infty \text{ for some } c > 0\}$,

$h_M = \{x \in w : I_M(cx) < \infty \text{ for all } c > 0\}$,

where I_M is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k),\ x = (x_k) \in t_{\mathcal{M}}$$

We consider t_M equipped with the Luxemburg norm

$$||x|| = \inf\left\{k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1\right\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\left\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\right\}$$

A Musielak-Orlicz function (M_k) satisfies the Δ_2 -condition if there exist constants a, K > 0 and a sequence c = $(c_k)_{k=1}^{\infty} \in l_+^1$ the positive cone such that the inequality

$M_k(2u) \leq KM_k(u) + c_k$

holds for all $k \in N$ and $u \in R_+$, whenever $M_k(u) \leq a$.

6. DIFFERENCE SEQUENCE SPACES

The notion of difference sequence spaces was introduced, who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and C¸

olakby introducing the spaces $l_\infty(\Delta m)$, $c(\Delta m)$ and $c_0(\Delta m)$. Later the concept have been studied. Another type of generalization of the difference sequence spaces studied the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Recently, it has introduced a new type of generalized difference operators and unified those as follows [14].

Let m, n be non-negative integers, then for $Z = l_\infty, c, c_0$ we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta_n^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+nv}.$$

Taking n = 1, we get the spaces which were studied by Et and C¸ olak. Taking m = n = 1, we get the spaces which were introduced and studied.

Similarly, we can define difference operators on double sequence spaces as:

$\Delta x_{k,l} = (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1})$

$= x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1},$

$\Delta^m x_{k,l} = \Delta^{m-1} x_{k,l} - \Delta^{m-1} x_{k,l+1} - \Delta^{m-1} x_{k+1,l} + \Delta^{m-1} x_{k+1,l+1}$

And

$$\Delta_n^m x_{k,l} = \Delta_n^{m-1} x_{k,l} - \Delta_n^{m-1} x_{k,l+1} - \Delta_n^{m-1} x_{k+1,l} + \Delta_n^{m-1} x_{k+1,l+1}.$$

7. BK-SPACE

A Banach sequence space (η, S) is called a BK-space if the topology S of η is finer than the co-ordinatewise convergence topology, or equivalently, the projection maps Pi : η → K, Pi(x) = xi , i ≥ 1 are continuous, where K is the scalar field R or C. For x = (x1, ...,xn, ...) and n ∈ N, we write the n th section of x as $x(n) = (x_1, ..., x_n, 0, 0, ...)$. If x (n) → x in (η, S) for each x ∈η, we say that (η, S) is an AK-space.

8. CONCLUSION

It is concluded that For a very long time, substitution operation which is basic to mathematics have been pursued and studies composition of functions. The study, as apart of operator theory of linear operators induced by composition with fixed function for example, iteration of rational functions has a relative short history. The research on composition operators is part of what can be called "concrete operator theory." In operator theory the normal operators are only class of operators on infinite dimensional. Although substitution operators have been studied on a variety of spaces. Moreover such spaces have played an important role in the development of analysis. Substitution operators establishes a contact with ergodic theory, entropy theory, classical mechanics, appear in topological dynamics. For a very long time, substitution operation which is basic to mathematics have been pursued and studies composition of functions. The study, as a part of operator theory of linear operators induced by composition with fixed function for example, iteration of rational functions has a relative short history. The research on composition operators is part of what can be called "concrete operator theory." In operator theory the normal operators are only class of operators on infinite dimensional. Although substitution operators have been studied on a variety of spaces. Moreover such spaces have played an important role in the development of analysis. Substitution operators establishes a contact with ergodic theory, entropy theory, classical mechanics, appear in topological dynamics.

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