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CONVERGENCE AND SUMMABILITY OF GENERAL ORTHOGONAL SERIES

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Abstract

A tremendous amount of work has been done in recent years in the field of summability, convergence and approximation problems of general orthogonal series. The theory of orthogonal series was originated during the discussion of the problem of vibrating string more than 200 years ago. There has been a huge amount of development in the convergence, summability and approximation problems of Fourier series. However, less attention has been paid in the theory of orthogonal series. In our work, we would like to discuss the recent trends in and particular orthogonal series.

Keyword: Orthogonal series, convergence, summability

1. INTRODUCTION

The problem of vibrating string was considered by Bernoulli D. (1700-1782) around 1753. The problem was to find the solution of the following partial differential equation with given initial and boundary conditions [1]:

$$u_{tt} \ = \ , u(x,0) \ = \ f(x), u_t\ (x,0) \ = \ 0, u(0,t) \ = \ u(\pi,t) \ = \ 0$$

This is the well-known wave equation. The graph of $(x,)$ represents the shape of vibrating string at time t. D'Alembert J. L. R. gave the solution of the form [2]

$$u(x,t) = \frac{1}{2} f(x+t) + \frac{1}{2} f(x-t)$$

We can say that $(x,)$ is a solution of the given problem if f is odd 2π periodic extension to the real line R of the given initial condition in the interval $[0, \pi]$.

Bernoulli D. suggested the solution of the form [4]

$$u(x,t) = \sum_{k=0}^{\infty} a_k \cos(kx) b_k \sin(kx)$$

Based on this observation Bernoulli D. believed the possibility of expanding an arbitrary periodic function (x) with $(0) = (\pi) = 0$ in terms of $\sin(kx)$ but he didn't have a clue how to calculate Fourier coefficients [5].

Later on Euler L. and Lagrange J. L. (1736-1813) also worked on the same problem. They have given the possibility of representing an arbitrary function by trigonometric series [6]. In 1807, while working on heat conduction Fourier J. (1768-1830) suggested the way for calculating Fourier coefficient and as a consequence a Fourier series of the function (x). This is how Fourier series was developed [7].

2. SUMMABILITY OF ORTHOGONAL SERIES

It was first shown by Kaczmarz, S. 1927 [3] that under the condition

$$\sum_{n=0}^{\infty} c_n^2 < \infty$$

the necessary and sufficient condition for general orthogonal series (1-38) to be $(C, 1)$ summable almost everywhere is that there exist a sequence of partial sums $\{S\upsilon n\ (x)\}$: $1 \leq q \leq \upsilon n + 1/\upsilon n \leq r$ convergent everywhere in the interval of orthogonality. The same result was extended by Zygmund, A. 1927 [14], Zygmund, A. 1959 [15], for (C, α), $\alpha > 0$ summability.

The classical result for $(C, 1)$ summability reads as follows;

The condition

$$\sum_{n=2}^{\infty} c_n^2 \log n < \infty$$

is sufficient for $(C, 1)$ summability of (1-38).

Again Borgen, S. 1928 [2], Kaczmarz, S. 1927 [3], Menchoff, D. 1925 [8], and Menchoff, D. 1926 [9], have refined the same condition and established an analogous of Rademacher-Menchoff theorem for (C, α), summability;

which shows that

$$\sum_{n=3}^{\infty} c_n^2 (log(logn))^2 < \infty$$

implies $(C, \alpha > 0)$ summability of series (1-38).

3. ABSOLUTE SUMMABILITY OF ORTHOGONAL SERIES: BANACH AND GENERALIZED NÖRLUND SUMMABILTY

The absolute summability of an orthogonal series has been studied by many mathematicians like Krasniqi, Xh. Z. 2010 [4], Krasniqi, Xh. Z. 2011 [5], Krasniqi, Xh. Z. 2012 [6], Leindler, L.1995 [7], Patel, D. 1996 [11], Shah, B. 1993 [12].

Paikray, S. et al. 2012 [10] have proved the following theorem:

Theorem 3.1

$\Psi\alpha\ (+0) = 0\ ,\ 0 < \alpha < 1$

And

$$\int_0^{\pi} \frac{d\Psi_\alpha(u)}{u^\alpha \log(n+U)} < \infty$$

then, the series

$$\sum_{n=1}^{\infty} \frac{B_n(t)}{log(n+1)}$$

is $|B|$ summable at t = x

if

$$\sum_{k\leq\frac{1}{u}} log(n+U)k^{\alpha-1} = O(U^{\alpha} log(n+2)) \quad ; U = \left[\frac{1}{u}\right]$$

It has proved the following theorem on Cesàrosummability of order α for orthogonal series

Theorem 3.2

Let {φn (x)} be orthonormal system defined in the interval (a, b) and let $\alpha > 0$. If the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}} \left[\sum_{k=1}^{n} k^2 (n-k+1)^{2(\alpha-1)} {a_k}^2\right]^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\alpha}}$$

converges, then the orthogonal series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x)$$

issummable $|C, \alpha|$ for almost every x.

Our theorem is as follows:

Theorem 3.3

Let $\{(x)\}$ be an orthonormal system defined in (a, b).

If

$$\sum_{k=1}^{\infty} \frac{1}{k+1} \left\{\sum_{v=1}^{k} c_{n+v}^2\right\}^{\frac{1}{2}} < \infty$$

for all n , then orthogonal series (1-28) is absolutely Banachsummable i.e. $|B|$ summable for every x .

it is obtained the following result on strong Nörlundsummability of orthogonal series.

4. MATRIX SUMMABILITY OF AN ORTHOGONAL SERIES

Based on definition of Krasniqi, Xh. Z. et. al. 2012 [6] has proved the following theorems:

Theorem 4.1

If the series

$$\sum_{n=1}^{\infty}\left\{n^{2-\frac{2}{k}}\sum_{j=0}^{n}\left|\hat{a}_{n,j}\right|^{2}\left|c_{j}\right|^{2}\right\}^{\frac{k}{2}}$$

converge for $1 \leq k \leq 2$, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|A|k$summable almost everywhere.

Theorem 4.2

Let $1 \leq \mathrm{k} \leq 2$ and $\{(n)\}$ be a positive sequence such that $\{\ (n)/n\ \}$ is non-increasing sequence and the series

$$\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$$

converges. If the following series

$$\sum_{n=1}^{\infty}|c_n|^2\Omega^{\frac{2}{k}-1}(n)\omega^{(k)}(A;n)$$

converges, then the orthogonal series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \in \ |A|_k$$

almost everywhere.

5. APPROXIMATION BY NÖRLUND MEANS OF AN ORTHOGONAL SERIES

It has studied the rate of approximation by Nörlund means for Walsh-Fourier series. He has proved the following theorem:

Theorem 5.1

Let $f \in \quad 1 \le p \le \infty, let\ n = 2m + k, 1 \le k \le 2m, m \ge 1\ and\ let\ \{q_k\ ;\ k \ge 0\}$ be a sequence of non-negative numbers such that

$$\frac{n^{\gamma-1}}{Q_n^{\gamma}} \sum_{k=0}^{n-1} q_k^{\gamma} = O(1)$$

for some $1 < \gamma \le 2$

If $\{q_k\}$ is non-decreasing, then

$$||t_n(f) - f||_p \le \frac{5}{2Q_n} \sum_{j=o}^{m-1} 2^j q_{n-2^j}\, \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\}$$

If $\{q_k\}$ is non-increasing then

$$||t_n(f) - f||_p \le \frac{5}{2Q_n} \sum_{j=o}^{m-1} \left(Q_{n-2^j+1} - Q_{n-2^{j+1}+1}\right) \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\}$$

It has proved the following theorem [13]:

Theorem 5.2

Let$f \in Lp, 1 \le p \le \infty, let\ n \coloneqq 2m + k, 1 \le k \le 2m, m \ge 1\ and\ let\ \{q_k\ ;\ k \ge 0\}$ be a sequence of non-negative numbers.

If $\{q_k\}$ is non-decreasing and satisfies the conditions [14]

$$\frac{np_n}{P_n} = O(1)$$

Then

$$||\bar{t}_n(f) - f||_p \le \frac{3}{P_n}\sum_{j=o}^{m-1} 2^j p_{2^{j+1}-1}\, \omega_p\left(f, 2^{-j}\right) + O\{\omega_p(f, 2^{-m})\}$$

If $\{p_k\}$ is non-increasing then

$$||\bar{t}_n(f) - f||_p \le \frac{3}{P_n}\sum_{j=o}^{m-1} 2^j p_{2^j}\, \omega_p\left(f, 2^{-j}\right) + O\{\omega_p(f, 2^{-m})\}$$

We have generalized the result for $(E, 1)$ summability.

6. CONCLUSION

It is concluded that since the 18th century, the research of L. Euler, D. Bernoulli, A. Legendre, P. Laplace, F. Bessel contains special orthonormal systems and expansion of a functions with respect to orthonormal systems in the subjects like mathematics, astronomy, mechanics, and physics. There was a remarkable progress of the theory of general orthogonal series during 20th century. Several researchers have made use of orthonormal systems of functions and orthogonal series in the areas like computational mathematics, functional analysis, mathematical physics, mathematical statistics, operational calculus, quantum mechanics, etc.

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