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A STUDY OF DIFFERENCE EQUATIONS AND OSCILLATORY BEHAVIOR OF THE SOLUTIONS OF SECOND ORDER NONLINEAR DELAY DIFFERENCE EQUATIONS

VED PRAKASH SARVAIYA

RESEARCH SCHOLAR, OPJS UNIVERSITY CHURU RAJASTHAN

DR. ASHWINI NAGPAL

ASSOCIATE PROFESSOR, OPJS UNIVERSITY CHURU RAJASTHAN

Abstract

The main theme of difference equations is that of recursion. Computations performed in a recurrent or repeated manner. In fact, difference equations are sometimes referred to as recursion relations. The theory of difference equations is interesting in itself and it has become significant for its various applications in numerical analysis, control theory, finite mathematics, statistics, economics, biology and computer science. For example, A gambler plays a sequence of games against an adversary. The probability that the gambler wins R 1 in any given game is q and the probability of him losing R 1 is 1 − q. He quits the game if he either wins a prescribed amount of N rands, or loses all his money; in the latter case we say that he has been ruined. Let p(n) denotes the probability that the gambler will be ruined if he starts gambling with n rands.

KEYWORDS: Difference Equations, Oscillatory Behavior, Second Order Nonlinear, Difference Equations

INTRODUCTION

In this chapter, we concerned with the oscillatory behavior of the solutions of second order nonlinear delay difference equations of the form

$$\Delta(a_n(\Delta u_n)) + q_n f(u_{\sigma(n)}) = 0,\ n \in \mathbb{N}_0, \qquad (2.1)$$

where Δ is the forward difference operator defined by

$\Delta u_n = u_{n+1} - u_n,\ \Delta^2 u_n = \Delta(\Delta u_n),\ \mathbb{N}_0 = \{n_0, n_0+1, n_0+2, \ldots\}$ and n0 is a nonnegative integer subject to the following conditions:

(C_1) $\{a_n\}$ is a positive sequence with $\sum_{s=n_0}^{\infty} \frac{1}{a_s} = \infty;$; and {qn}is

non-negative real sequence having a positive subsequence.

(C2) f : $\mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function

such that $uf(u) > 0$ and $\frac{f(u)}{u} \geq L > 0, \text{for } u \neq 0$.

(C3) $\{\sigma(n)\}$ is a sequence of positive integers such that

$\sigma(n) \leq n$ and $\lim_{n\to\infty} \sigma(n) = \infty.$

(C4) $R_n = \sum_{s=n_1}^{n-1} \frac{1}{a_s} \to \infty$ as $n \to \infty$.

By a solution of equation (2.1) we mean a real sequence {un} which satisfies equation (2.1) for all n$\in\mathbb{N}$ 0. We consider only that solution {un} of equation (2.1) which satisfies sup $\{|u_n| : n \geq N\} > 0$ for all $n \in \mathbb{N}_0$. A solution of equation (2.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and non-oscillatory otherwise.

BOUNDED OSCILLATION OF DELAY DIFFERENCE EQUATIONS

In this section we establish sufficient conditions for the oscillations of all solutions of equation (2.1). To prove the main result, we need the following Lemma.

Lemma 2.2.1

Assume that (C1)-(C3) hold. If (un) is a positive solution of the equation (2.1), then

$$u_n > 0,\ a_n\Delta u_n > 0,\ \Delta(a_n\Delta u_n) < 0, \qquad (2.2)$$

eventually.

Proof

Assume that {un}is a positive solution of equation (2.1).

un > 0 for n > n1 > n .0 From the equation (2.1),

suppose un < 0 Now (2.1) can be written as

$$\Delta\left(a_n(\Delta u_n)\right) \le -Lq_n u_{\sigma(n)}, \quad \text{for some } L > 0. \qquad (2.3)$$

Consequently, $a_n\Delta u_n$ is non-increasing and either $a_n\Delta u_n > 0$ or

$a_n\Delta u_n \le 0$. If $a_n\Delta u_n \le 0$, then for $n \ge n_1$, we have

$$a_n\Delta u_n \le a_{n_1}\Delta u_{n_1} < 0. \qquad (2.4)$$

Dividing the last inequality by an1 and then summing the resulting inequality from n to n - 1, we obtain

$$u_n < u_{n_1} + a_{n_1}\Delta u_{n_1}\left(L\sum_{s=n_1}^{n-1}\frac{1}{a_s}\right) \to -\infty \text{ as } n \to \infty, \qquad (2.5)$$

which is contradiction for the positivity of {un}of the equation (2.1).

This completes the proof.

Next we consider the equation (2.1) with qn < 0 for all n > □ (n0). For the sake of convenience, we write (2.1) in the

$$\Delta(a_n\Delta u_n) = Q_n f\left(u_{\sigma(n)}\right),\ n \ge \square(n_0), \qquad (2.6)$$

where $Q_n = -q_n$ for all $n \ge \square(n_0)$.

Theorem 2.2.1

Assume that (C1)-(C3) hold. If $\limsup_{n\to\infty} \sum_{s=\sigma(n)}^{n} \frac{1}{a_s}\sum_{t=s}^{n-1} Q_t > \frac{1}{L}$, then every bounded solution of the equation (2.1) is oscillatory.

Proof

Without the loss of generality, we may assume that {un} is a bounded and eventually positive solution of equation (2.6). Then from equation (2.6), we have $\Delta(a_n\Delta u_n) \ge 0$ for all $n \ge \square(n_0)$.

By the boundedness of $\{u_n\}$, we have $\Delta u_n < 0$ eventually, $u_n > 0$ for $n \ge n_1 \in \square(n_0)$. In view of (C2), equation (2.6) can be written as

$$\Delta(a_n\Delta u_n) \ge L Q_n u_{\sigma(n)}, \qquad n \ge \square(n_0) \qquad (2.7)$$

summing (2.7) from s to n we have,

$$-\Delta u_s \ge u_{\sigma(n)}\frac{L}{a_s}\sum_{t=s}^{n} Q_t$$

again summing from $\sigma(n)$ to n, we see that

$$-u_n + u_{\sigma(n)} \geq L u_{\sigma(n)} \sum_{s=\sigma(n)}^{n} \frac{1}{a_s} \sum_{t=s}^{n} Q_t, \qquad (2.8)$$

we claim that limun = 0. From (2.8), we have

$$\limsup_{n\to\infty} \sum_{s=\sigma(n)}^{n} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t \leq 0.$$

which is a contradiction.

Hence there exists an integer n > (n0) such that $u_{\sigma(n)} < 1$ for $n \geq \square (n_0)$. Then from (2.8) we have

$$u_n + u_{\sigma(n)} \left[L \sum_{s=\sigma(n)}^{n} \frac{1}{a_s} \sum_{t=s}^{n} Q_t - 1 \right] \leq 0. \qquad (2.9)$$

From the equation (2.6) $\sum_{s=\sigma(n)}^{n} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t = \infty.$ Hence (2.7) implies that limun □ □, which is contrary to the boundness of {un}. This completes the proof of our theorem.

EXISTENCE OF CLASS M SOLUTION OF DELAY DIFFERENCE EQUATIONS

In this section, we use a fixed-point theorem to prove the existence of solutions of the non-linear equation (2.1). From results of Cheng,et.,al.,[27], it is known that any nontrivial solution {un} of (2.1) is non-oscillatory and belongs to Class M

$$M = \left\{ \begin{array}{l} \{u_n\} : \text{there exits an integer } n \in \square (n_0) \\ \text{such that } u_n \Delta u_n < 0 \text{ for all } n \geq \square (n_o) \end{array} \right\}.$$

Define

$$M_0 = \left\{\{u_n\} \in M : \lim_{n\to\infty} u_n = 0\right\} \text{and } M_L = \left\{\{u_n\} \in M : \lim_{n\to\infty} u_n \neq 0\right\}. \qquad (2.10)$$

Let $E_A = \sum_{s=0}^{\infty} \frac{1}{a_s}$ and $E_Q = \sum_{s=0}^{\infty} Q_s$, where $Q_n = -q_n$ for all $n \geq \square (n_0)$.

Theorem 2.3.1

Let $E_A < \infty$ and $E_Q < \infty$. . Then, equation (2.1) has at least one solution in the class M0 and at least one solution in the class ML .

Proof

First, we prove the existence of a positive decreasing solution of (2.1).

Let K $K = \max\left\{|f(u)| : \frac{2}{3} \leq u \leq \frac{4}{3}\right\}$ and choose n0 large enough so that

$$K\left[\left(\sum_{s=N(n_0)}^{\infty} \frac{1}{a_s}\right)\sum_{s=n_0}^{\infty} Q_s + \sum_{s=N(n_0)}^{\infty} \frac{1}{a_s}\sum_{s=n}^{\infty} Q_s\right] < \frac{1}{3}, \qquad (2.11)$$

$$\sum_{s=N(n_0)}^{\infty} \frac{1}{a_s} < \frac{1}{3}. \qquad (2.12)$$

And

Consider the Banach Space BN(n)0 of all real sequences

$U = \{u_n\}, n \geq N(n_0)$, with the supremum norm

$$\|u\| = \sup_{n \geq N(n_0)} |u_n|, \text{ and let } S = \left\{U \in B_{N(n_0)} : \frac{2}{3} \leq u_n \leq \frac{4}{3}, n \geq N(n_0)\right\}.$$

Clearly, S is a bounded, convex and closed subset of BN(n)0 .

We define an operator T : S □ BN(n)0 by

$$Tu_n = 1 + \left[\sum_{s=N(n_0)}^{\infty} \frac{1}{a_s} - \sum_{s=N(n_0)}^{n-1} Q_s \left(\sum_{t=N(n_0)}^{\infty} \frac{1}{a_t} \right) f\left(u_{\sigma(n)}\right) \right.$$

$$\left. - \left(\sum_{s=N(n_0)}^{n-1} \frac{1}{a_s} \right) \sum_{s=n}^{\infty} Q_s f\left(u_{\sigma(n)}\right) \right], \quad n \geq N(n_0). \qquad (2.13)$$

Next we show that T satisfies the condition of Schauder's fixed-point theorem.

(i) T maps S into itself. In fact, if U□S, then from (2.12) and (2.13),

we have

$$Tu_n \geq 1 - K\left[\left(\sum_{s=N(n_0)}^{\infty} \frac{1}{a_s} \right) \sum_{s=n_0}^{\infty} Q_s + \sum_{s=N(n_0)}^{\infty} \frac{1}{a_s} \sum_{s=n}^{\infty} Q_s \right] \geq 1 - \frac{1}{3} = \frac{2}{3}, \qquad (2.14)$$

and from (2.12) we have

$$Tu_n \leq 1 + \left(\sum_{s=N(n_0)}^{\infty} \frac{1}{a_s} \right) \leq 1 + \frac{1}{3} = \frac{4}{3}. \qquad (2.15)$$

Therefore, $T(S) \subset S$.

(ii) T is continuous, Let $u^{(i)} = \{u_n^{(i)}\} \in S$, , such that

$$\{u_n^{(i)}\} \to \{u\} \text{ as } i \to \infty. \qquad (2.16)$$

Because S is closed. Let $U = \{u_n\} \in S$ for $n \geq N(n_0)$. . Let $\varepsilon > 0$ be given, and choose n large enough so that

$$\max\left\{ M \sum_{n=N(n_0)}^{\infty} Q_n, 2M \sum_{s=N(n_0)}^{\infty} Q_n \left(\sum_{t=n_0}^{\infty} \frac{1}{a_t} \right) \right\} < \varepsilon. \qquad (2.17)$$

We have

$$\left|Tu_n^{(i)} - Tu_n\right| \le \sum_{s=N(n_0)}^{\infty} Q_s \left(\sum_{s=N(n_0)}^{\infty} \frac{1}{a_s} \right) \left| f\left(u^i_{\sigma(n)}\right) - f\left(u_{\sigma(n)}\right) \right| + \frac{5M}{3} \sum_{s=N(n_0)}^{n-1} Q_s. \quad (2.18)$$

In view of (2.16) and the continuity of f, we have

$$\left| f\left(u^i_{\sigma(n)}\right) - f\left(u_{\sigma(n)}\right) \right| \to 0 \text{ as } i \to \infty.$$

It follows that

$$\lim_{n\to\infty} \left\| Tu_n^{(i)} - Tu_n \right\| = 0. \quad (2.19)$$

Hence T is Continuous.

(iii) Next, we shall show that T(S) is relatively compact. As proved by Cheng and Patula [Theorem 3.3,[28]]. It suffices to show that T(S) is uniformly Cauchy. Let $U = \{u_n\} \in S$ for $n \ge N(n_0)$.

Then

$$\left|Tu_j - Tu_n\right| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad (2.20)$$

From the hypothesis it is clear that for a given $\varepsilon > 0$, , there exists an integer $n \ge n_1 \ge N(n_0)$. Then T(S) is uniformly Cauchy. Hence T(S) is relatively compact. Hence by Schauder's fixed-point Theorem [1.6.8] there exists $U \in S$ such that $TU = U$ and

$$\Delta u_n = \left[Lu_{\sigma(n)} \sum_{s=\sigma(n)}^{n} \frac{1}{a_s} \sum_{t=s}^{n} Q_t - 1 \right] < 0.$$

It is easy to see that un is solution of (2.1) that is

$$u_n = 1 + \left[\sum_{s=N(n_0)}^{\infty} \frac{1}{a_s} \left(\sum_{s=N(n_0)}^{\infty} Q_s f\left(u_{\sigma(n)}\right) \right) \right], \quad n \in N(n_0)$$

And $\left\{ \frac{1}{3} \le u_n \le \frac{4}{3} \right\},$ we see that { u} is an eventually positive

decreasing solution of (2.1) with $\lim_{n\to\infty} u_n = \ell \ne 0.$ Hence $M_L \ne \varnothing.$.

(iv) Next we prove the existence of eventually positive decreasing solution (2.1) that tends to zero as $n \to \infty.$.

Let $K = \max\left\{ \left|f(u)\right| : 0 \le u_n \le \frac{1}{3} \right\}$ and choose $N(n_0)$ such that

$$K \sum_{s=N(n_0)}^{\infty} \frac{1}{a_s} \left(\sum_{s=n}^{\infty} Q_s \right) \le \frac{1}{3}. \qquad (2.21)$$

Let BN(n)0 be the Banach space defined above,

Let $S = \left\{ U \in B_{N(n_0)} : 0 \le u_n \le \frac{1}{3}, n \ge N(n_0) \right\},$ and define the operator T by

$$Tu_n = \sum_{s=N(n_0)}^{\infty} \frac{1}{a_s} \sum_{s=n}^{\infty} Q_s f\left(u_{\sigma(s)}\right), n \ge N(n_0). \qquad (2.22)$$

Now, it is easy to see that the operator T satisfies the assumptions of Schauder's fixed-point theorem.

Therefore, there exists an $U \in S$ such that $TU = U,$ that is,

$$\Delta u_n = -\frac{1}{a_n} \left(\sum_{s=N(n_0)}^{\infty} Q_s f\left(u_{\sigma(s)}\right) \right) < 0, \qquad (2.23)$$

and so the solution of the equation (2.1) is

$$u_n = \sum_{s=N(n_0)}^{\infty} \frac{1}{a_n} \left(\sum_{s=N(n_0)}^{\infty} Q_s f\left(u_{\sigma(s)}\right) \right) \text{ and } \quad u_n \to 0 \text{ as } n \to \infty .$$

Then we see that $\{u_n\} \in M_0 \neq \varnothing$. This completes the proof of the theorem.

CONCLUSION

The research work reported in this thesis deals with the oscillatory and asymptotic properties of second order neutral advanced difference equations with variable coefficients. We have obtained sufficient conditions for oscillation and asymptotic behavior of all solutions of the equations under various conditions on the coefficients.

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