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STUDYING FRACTIONAL GENERALIZED INTEGRAL TRANSFORM AND THEIR IMPORTANT PROPERTIES

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ABSTRACT

Due to the possession of the non-local property of the fractional derivative, the notion of Caputo type fractional derivatives are introduced in G-transform and a novel technique 'Fractional Generalized Homotopy Analysis Method (FGHAM)' is proposed using Mittag-Leffler function and its standard properties are studied. The performance of the proposed method is analyzed by solving various categories of nonlinear fractional differential equations like NavierStokes's model and Riccatti equations. The closed form series solutions obtained using FGHAM show an excellent agreement with the other existing methods and the results are statistically validated.

Further, FGHAM is applied to solve the fractional Black-Scholes model in financial mathematics. The convergence of the series solutions is well established using Cauchy criteria. Using the numerical example, an analytical expression for the implied volatility is derived and the non-local behavior is studied for the various values of the fractional parameter and the results are statistically validated. The solutions obtained using FGHAM are compared with the other existing methods, including Monte-Carlo simulation and observed to have a good agreement. The implementation of FGHAM overcomes the difficulty of tackling iterative differentiation and integration in solving the fractional nonlinear differential equations.

Keywords: - FGHAM, Equation, Fractional, Theorem,Homotopy

I. INTRODUCTION

'Fractional Generalized Homotopy Analysis Method' (FGHAM) which is developed by incorporating the fractional derivatives in the generalized integral transform method. The Homotopy analysis method is combined with the fractional generalized integral transform method to solve the fractional order nonlinear differential equations. The performance of the proposed method is analyzed by solving various categories of nonlinear fractional differential equations like NavierStokes's model and Riccati equations etc.

Unlike the other analytical methods, the hybrid method provides a better way to control the convergence region of the obtained series solution through an auxiliary parameter *h*. Further, as proposed in this paper, the FGHAM along with the several examples reveal that this method can be effectively used as a tool for solving various kinds of nonlinear fractional differential equations.

II. REVIEW OF LITERATURE

JAVID ET AL. (2019) applied the generalized unified method to analyze the thermophoretic motion equation. They analyzed the soliton-like thermophoresis of wrinkles in graphene sheet based on the Korteweg-de Vries (KdV) equation and illustrated the solutions graphically. Osman *et al.* (2019c) applied the modified auxiliary equation method to investigate the complex wave structures related to the Ginzburg–Landau model. Lu *et al.* (2020) examined the Cahn–Hilliard equation to study the physical behaviors of the basic elements related to the phase decomposition of (Fe–Cr–Mo) and, (Fe–Cr–Cu) ternary alloys using both the analytical and numerical methods. Further, they studied the relevant dynamical separation process.

ZHAO AND XU (2015) investigated a new delay type of integral inequality by using the properties of the modified Riemann-Liouville fractional derivative. Recently, Khan *et al.* (2010) developed a Laplace decomposition method(LDM) using adomian polynomials and applied the method to solve the nonlinear coupled partial differential equations. Gondal and Khan (2010) proposed a Homotopy Perturbation Transform Pade Method (HPTPM) to obtain the series solution of the nonlinear exponential boundary layer equation. Khan and Gondal (2011) introduced the Modified Laplace Pade Decomposition Method (MLPDM), which is a combination of modified Laplace decomposition and Pade approximation, to provide an approximate analytical solution to the Thomas-Fermi equation. Khan *et al.* (2011) presented a Variational Iterative Pade Method (VIPM) to investigate the two dimensional exponential stretching sheet problem. Jafari*et al.* (2011) proposed a Two-Step Adomian Decomposition Method (TSADM) as a modification over an ADM (Adomian Decomposition Method), in order to obtain the series solution for both linear and nonlinear equations.

GONDAL ET AL. (2011) applied a Homotopy Analysis Transform Method (HATM), which is a combination of HAM and LDM, to effectively solve the linear and nonlinear partial differential equations with less computation. Khan and Gondal (2012) constructed a new Two-Step Laplace Decomposition Algorithm (TSLDA) which is a combination of the Laplace transform and decomposition method for the solution of Abel's type singular integral equations. A modified Laplace decomposition method for the analytical solution for the eighthorder boundary value problem was also suggested. Gondal*et al.* (2013) presented the homotopy analysis transform method to find the solution of fractional diffusion-type equations.

MERDAN (2012) proposed the Fractional Variational Iteration Method (FVIM) to resolve the nonlinear fractional reaction-diffusion with the modified Riemann-Liouville derivative. The solutions obtained are reliable, and the FVIM is an effective method for solving nonlinear partial differential equations with the modified Riemann-Liouville derivative. Khan *et al.* (2012) suggested the Fractional Homotopy Perturbation Method (FHPM), which is based on He's homotopy perturbation method via the modified Riemann–Liouville derivative, for analyzing the fractional differential equations of any fractional parameter. Khan (2014) provided an approximate analytical solution to the 12th order differential equation by using homotopy analysis transform method. Zheng and Feng (2014) examined the space fractional KdV equation and the space-time fractional Fokas equation by using the concept of the modified Riemann-Liouville derivative. Bagyalakshmi and SaiSundara Krishnan (2020) presented a hybrid technique to solve the nonlinear fractional partial differential equations, which is a combination of the Tarig transform and Projected Differential Transform Method (TPDTM).

III. MOTIVATION OF THE WORK

Recently, many researchers are attracted towards the theoretical studies of various integral transforms due to its wide range of applications in solving differential equations. The computational methodology which is based on integral transforms can be used effectively for solving especially linear fractional differential equations. In order to solve the nonlinear fractional differential equations, several authors have adopted mathematical techniques like homotopy analysis and homotopy perturbation method etc. Ragab*et al.* (2012) applied homotopy analysis method (HAM) using Caputo fractional derivatives to solve the nonlinear fractional Navier-Stokes equation and then obtained the closed form series solution. The auxiliary parameter 'h' present in the HAM provided a simple way to adjust and control the convergence region and rate of convergence of the series solution.

Morales-Delgado *et al.* (2016) analyzed various fractional partial differential equations using Laplace HAM. Saad and Al-Shomrani (2016) presented the Homotopy Analysis Transform Method (HATM) for solving the fractional order Riccati differential equations.

Motivated by the above mentioned techniques, a new fractional generalized integral transform (G - Transform) using Caputo derivative and the Mittag - Leffler function of fractional order was introduced by Loonker and Banerji (2017) for solving the fractional differential equations.

IV. FRACTIONAL GENERALIZED INTEGRAL TRANSFORM (FRACTIONAL G - TRANSFORM)

This section discusses some of the properties of the fractional ${}^{C}G_{\alpha}$ − transform which is introduced using the Mittag− Leffler function and Caputo fractional derivative sense as given in Definition 3.1.

Definition Let $g(t)$ be any time domain function defined for $t > 0$. Then, the fractional G − Transform of order α of $g(t)$ in Caputo sense is denoted by ${}^{C}G_{\alpha}[g(t)]$ and is defined as

$$ {}^{C}G_{\alpha}[g(t)] = H_{\alpha}[u] = u^{p+1}\int_{0}^{\infty} g(ut)E_{\alpha}(-t)^{\alpha}(dt)^{\alpha} $$

$$ = u^{p-\alpha+1}\int_{0}^{\infty} g(t)E_{\alpha}\left(\frac{-t}{u}\right)^{\alpha}(dt)^{\alpha} $$

$$ = \lim_{M\to\infty} u^{p-\alpha+1}\int_{0}^{M} g(t)E_{\alpha}\left(\frac{-t}{u}\right)^{\alpha}(dt)^{\alpha} $$

where E_{α} is the Mittag− Leffler function.

Theorem (Duality in Fractional ${}^{C}G_{\alpha}$ Transform)

If the fractional order Laplace Transform of a function $g(t)$ is $L_{\alpha}\{g(t)\} = F_{\alpha}(s)$, then the fractional ${}^{C}G_{\alpha}$ Transform of order α of $g(t)$ in Caputo sense is

$$ {}^{C}G_{\alpha}[g(t)] = H_{\alpha}(u) = u^{p-\alpha+1}F_{\alpha}\left(\frac{1}{u}\right). \qquad (3.4) $$

Proof:

$$ {}^{C}G_{\alpha}[g(t)] = H_{\alpha}[u] = u^{p+1}\int_{0}^{\infty} g(ut)E_{\alpha}(-t)^{\alpha}(dt)^{\alpha} $$

$$ = \lim_{M\to\infty} \alpha u^{p+1}\int_{0}^{M} (M-t)^{\alpha-1}g(ut)E_{\alpha}(-t)^{\alpha}dt \qquad (3.5) $$

Put $ut = x$ in ,

$$ {}^{C}G_{\alpha}[g(t)] = \lim_{M\to\infty} \alpha u^{p+1} \int_0^{\frac{m}{u}} \left(M - \frac{x}{u}\right)^{\alpha-1} g(z) E_{\alpha} \left(\frac{-x}{u}\right)^{\alpha} \frac{dx}{u} $$

$$ = \lim_{M\to\infty} \alpha u^{p+1} \int_0^{\frac{m}{u}} (Mu - x)^{\alpha-1} g(z) E_{\alpha} \left(\frac{-x}{u}\right)^{\alpha} \frac{dx}{u^{\alpha}} $$

By using Laplace Transform,

$$ {}^{C}G_{\alpha}[g(t)] = H_{\alpha}(u) = u^{p-\alpha+1} F_{\alpha}\left(\frac{1}{u}\right). $$

V. CONCLUSION

The notion of Fractional Generalized Integral Transform and their important properties were studied. The fractional G− Transform was combined with HAM and this hybrid method was implemented to solve the nonlinear fractional differential equations. The proposed method avoided multiple differentiation and integration in solving the fractional differential equations and obtained the solution by the suitable selection of auxiliary parameters. Further, the solution curves were also graphically illustrated for the various fractional values.

REFERENCES

1. Cheng, J 2008, 'Application of the homotopy analysis method in nonlinear mechanics and finance', PhD dissertation, Shanghai Jiao Tong University.

2. Cheng, J, Zhu, SP & Liao, SJ 2010, 'An explicit series approximation to the optimal exercise boundary of American put options', Communications in Nonlinear Science and Numerical Simulation, vol. 15, no. 5, pp. 1148–1158.

3. Company, R, Navarro, E, Pintos, JR &Ponsoda, E 2008, 'Numerical solution of linear and nonlinear Black–Scholes option pricing equations', Computers and Mathematics with Applications, vol. 56, pp. 813-821.

4. Dadachanji, Z 2015, 'Derivation of the Local Volatility Function, FX Barrier Options: A Comprehensive Guide for Industry Quants', Applied Quantitative Finanace, Palgrave Macmillan.

5. Dogan, N & Akin, O 2012, 'Series solution of epidemic model', Turkic World Mathematical Society Journal of Pure and Applied Mathematics, vol. 2, no. 2, pp. 238–244.

6. Elbeleze, AA, Kilicman, A &Taib, BM 2013, 'Homotopy perturbation method for fractional Black-Scholes European option pricing equations using Sumudu transform', Mathematical Problems in Engineering, Article ID. 524852.

7. Farhadi, A, Salehi, M &Erjaee, GH 2018, 'A new version of Black–Scholes equation presented by time-fractional derivative', Iranian Journal of Science and Technology, Transactions of Electrical Engineering, vol. 42, pp. 2159–2166.

8. Garven, JR 1986, 'A pedagogic note on the derivation of the Black-Scholes option pricing formula', Financial Review, vol. 21, no. 2, pp. 337–348.

9. Ghanbari, B, Osman, MS &Baleanu, D 2019, 'Generalized exponential rational function method for extended Zakharov–Kuzetsov equation with conformable derivative', Modern Physics Letters A, vol. 34, no. 20, Article ID. 1950155. doi:10.1142/s0217732319501554

10. Ghandehari, MAM &Ranjbar, M 2014, 'European option pricing of fractional Black-Scholes model with new Lagrange multipliers', Computational Methods for Differential Equations, vol. 2, no. 1, pp. 1–10.