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STUDYING NEW FRACTIONAL INTEGRAL INEQUALITIES

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ABSTRACT

Integral inequalities make up a comprehensive and prolific field of research within the field of mathematical interpretations. Integral inequalities in association with convexity have a strong relationship with symmetry. Different disciplines of mathematics and applied sciences have taken a new path as a result of the development of new fractional operators. Different new fractional operators have been used to improve some mathematical inequalities and to bring new ideas in recent years. To take steps forward, we prove various Grüss-type and Chebyshev-type inequalities for integrable functions in the frame of non-conformable fractional integral operators. The key results are proven using definitions of the fractional integrals, well-known classical inequalities, and classical relations.

Keywords: -Grüss- Type, Chebyshev- Type, Fractional, Calculus, Inequality.

I. INTRODUCTION

Fractional calculus theory gained popularity and was employed as a mathematical tool in a variety of pure and practical fields. This approach has previously been used in a variety of industries with some impressive results. It has been used in medicine, physics, modelling of diseases, nanotechnology, fluid mechanics, bioengineering, epidemiology, economics, and control systems. In applied mathematics, inequalities and their applications are crucial. Various fractional operators were used to show a collection of integral inequalities and their generalizations. To follow this trend, we use a generalized non-conformable fractional integral operator to show an improved version of the Grüss-type inequality. G. Grüss presented the well-known Grüss-type inequality in 1935, which was linked to the Chebyshev's inequality; see.

$$\left|\frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{S}(\mathrm{u})\mathfrak{Z}(\mathrm{u})\mathrm{du}-\left(\frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{S}(\mathrm{u})\mathrm{du}\right)\left(\frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{Z}(\mathrm{u})\mathrm{du}\right)\right|\leq\frac{(\mathcal{B}-\mathcal{A})(\mathcal{D}-\mathcal{C})}{4}.$$

Provided that S and Z are two integrable functions on [$, κ], satisfying the condition,

$$\mathcal{A}\leq\mathfrak{S}(\mathrm{u})\leq\mathcal{B},\;\; \mathcal{C}\leq\mathfrak{Z}(\mathrm{u})\leq\mathcal{D},\;\; \mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in\mathcal{R},\;\; \mathrm{u}\in[\varrho,\kappa].$$

For integrable functions, various types of inequalities have been established, but the Grüss inequality has been the focus of many studies as many scholars have examined it extensively. Chaos, bio-sciences, fluid dynamics, engineering, meteorology, biochemistry, vibration analysis, aerodynamics, and many other scientific fields benefit from this inequality. See for a steady growth of interest in such a field of study to address the difficulties of various applications of these variants.

$$T(\mathfrak{S},\mathfrak{Z})=\frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{S}(\mathrm{u})\mathfrak{Z}(\mathrm{u})du-\left(\frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{S}(\mathrm{u})du\right)\left(\frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{Z}(\mathrm{u})du\right),$$

where S and Z are two integral functions that are synchonous on [$, κ], given as

$$(\mathfrak{S}(\mathrm{u})-\mathfrak{S}(\mathrm{y}))(\mathfrak{Z}(\mathrm{u})-\mathfrak{Z}(\mathrm{y}))\geq 0,$$

for any u, y ∈ [$, κ]; then, the Chebyshev inequality states that T(S, Z) ≥ 0.

II. PRELIMINARIES

Definition 1

For each $\mathfrak{S}\in\mathcal{L}[\xi,\vartheta]$ *and* $0<\xi<\vartheta,$ we define $\mathrm{N}_{3}\mathfrak{J}_{\mathrm{u}}^{\eta}\mathfrak{S}(\mathrm{x})=\int_{\mathrm{u}}^{\mathrm{x}}\vartheta^{-\eta}\mathfrak{S}(\vartheta)d\vartheta,$

for every x, u ∈ [ξ, ϑ] and η ∈ R.

III. FRACTIONAL INEQUALITY OF GRÜSS TYPE

In this section, first, we prove some new integrable equalities; then, using these equalities and the Cauchy–Schwarz inequality, our main findings are presented.

Lemma 1. Let the integrable function on (0, ∞) be S with A, B ∈ R; then, ∀w > 0 and η > 0, the following equality holds true:

$$\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}^2(\mathtt{w})+\left({}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w})\right)^2$$
$$=\left(\mathcal{B}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w})-\mathcal{A}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}\right)-\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}((\mathcal{B}-\mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w})-\mathcal{A})).$$

Proof.

Let A, B ∈ R and S be an integrable function on $(0,\infty)\ \forall\ \mu,\rho\ \in\ (0,\infty);$ then, we have

$$\begin{aligned}&(\mathcal{B}-\mathfrak{S}(\rho))(\mathfrak{S}(\mu)-\mathcal{A})+(\mathcal{B}-\mathfrak{S}(\mu))(\mathfrak{S}(\rho)-\mathcal{A})\\&-(\mathcal{B}-\mathfrak{S}(\mu))(\mathfrak{S}(\mu)-\mathcal{A})-(\mathcal{B}-\mathfrak{S}(\rho))(\mathfrak{S}(\rho)-\mathcal{A})\\&=\mathfrak{S}^2(\mu)+\mathfrak{S}^2\rho+2\mathfrak{S}(\mu)\mathfrak{S}(\rho).\end{aligned}$$

If we multiply both sides of (6) by (w − μ) η−1 and integrate the resultant equality with respect to μ, we obtain

$$\begin{aligned}&(\mathcal{B}-\mathfrak{S}(\rho))\left({}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w})-\mathcal{A}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}\right)+\left(\mathcal{B}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w})\right)(\mathfrak{S}(\rho)-\mathcal{A})\\&-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}((\mathcal{B}-\mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w})-\mathcal{A}))-(\mathcal{B}-\mathfrak{S}(\rho))(\mathfrak{S}(\rho)-\mathcal{A})\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}\\&={}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}^2(\mathtt{w})+\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}\mathfrak{S}^2(\rho)+2\mathfrak{S}(\rho){}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w}).\end{aligned}$$

Upon multiplication of both sides of (7) by (w − ρ) η−1 and integration of the resultant equality with respect to ρ, we yield

$$\begin{aligned}&\left({}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w})-\mathcal{A}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}\right)\int_a^{\mathtt{w}}(\mathtt{w}-\rho)^{\eta-1}(\mathcal{B}-\mathfrak{S}(\rho))d\rho\\&+\left(\mathcal{B}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w})\right)\int_a^{\mathtt{w}}(\mathtt{w}-\rho)^{\eta-1}(\mathfrak{S}(\rho)-\mathcal{A})d\rho\\&-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}((\mathcal{B}-\mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w})-\mathcal{A}))\int_a^{\mathtt{w}}(\mathtt{w}-\rho)^{\eta-1}(\mathcal{B}-\mathfrak{S}(\rho))d\rho\\&-\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}\int_a^{\mathtt{w}}(\mathtt{w}-\rho)^{\eta-1}(\mathcal{B}-\mathfrak{S}(\rho))(\mathfrak{S}(\rho)-\mathcal{A})d\rho\\&=\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}^2(\mathtt{w})+\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}^2(\mathtt{w})+2{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w}),\end{aligned}$$

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This led us to the proof of Lemma

IV. CERTAIN NEW FRACTIONAL INTEGRAL INEQUALITIES

Here, we present some new type of inequalities (Theorems 4–6) pertaining to nonconformable fractional integral operator

Theorem 2

Let the positive functions defined on [0, ∞) be S and Z. Then, the inequalities holds:

1. $\frac{1}{p}\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})^p+\frac{1}{q}\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})^q\geq\left[\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}\right]^{-1}\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z}).$

2. $\frac{1}{p}\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})^p\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})^p+\frac{1}{q}\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})^q\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})^q\geq\left(\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})\right)^2.$

3. $\frac{1}{p}\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})^p\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})^q+\frac{1}{q}\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})^q\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})^p\geq\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z}^{p-1})\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z}^{q-1}).$

4. $\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})^p\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})^q\geq\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})\mathrm{N}_3\mathfrak{J}^{\eta}_{\varrho^+}\left(\mathfrak{S}^{p-1}\mathfrak{Z}^{q-1}\right),\ \textit{where for}\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1,$

Proof. From Young's inequality, we have

$$\frac{1}{p}\mathrm{u}^p+\frac{1}{q}v^q\geq\mathrm{uv},\quad\text{for all}\quad\mathrm{u},\mathrm{v}\geq 0,\ \ p,q>1,\ \ \frac{1}{p}+\frac{1}{q}=1.$$

If we choose u = S(μ) and v = Z(ρ), μ, ρ > o, the

$$\frac{1}{p}(\mathfrak{S}(\mu))^p+\frac{1}{q}(\mathfrak{Z}(\rho))^q\geq\mathfrak{S}(\mu)\mathfrak{Z}(\rho),\ \ \forall\ \mathfrak{S}(\mu)\mathfrak{Z}(\rho)\geq 0.$$

Multiplication of inequality (21) by (w − μ) η−1 , and integrating the resultant inequality with respect to μ, we get

$$\frac{1}{p}\int_a^{\mathtt{w}}(\mathtt{w}-\mu)^{\eta-1}(\mathfrak{S}(\mu))^p d\mu+\frac{1}{q}\mathfrak{Z}(\rho)^q\int_a^{\mathtt{w}}(\mathtt{w}-\mu)^{\eta-1}d\mu\geq\mathfrak{Z}(\rho)\int_a^{\mathtt{w}}(\mathtt{w}-\mu)^{\eta-1}\mathfrak{S}(\mu)d\mu.$$

Consequently

$$\frac{1}{p} {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}(\mathtt{w}))^p + \frac{(\mathtt{w}-\eta)^{\eta}}{q\eta}\mathfrak{Z}(\rho)^q \geq \mathfrak{Z}(\rho) {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}(\mathtt{w})).$$

Analogously, multiplying inequality (23) by (w − ρ) η−1 and integrating the obtained identity, we obtain

$$\frac{(\mathtt{w}-\eta)^{\eta}}{p\eta} {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}(\mathtt{w}))^p + \frac{(\mathtt{w}-\eta)^{\eta}}{q\eta} {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z}(\mathtt{w}))^q \geq {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}) {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w}).$$

This readily follows:

$$\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}\left[\frac{1}{p} {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}(\mathtt{w}))^p + \frac{1}{q} {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z}(\mathtt{w}))^q\right] \geq {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}) {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w}).$$

Additionally,

$$\frac{1}{p} {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}(\mathtt{w}))^p + \frac{1}{q} {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z}(\mathtt{w}))^q \geq \left[\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}\right]^{-1} {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}) {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w}),$$

which implies (i). Similarly, we can prove the rest of the inequalities by making the correct choice of parameters as follows:

$$U = \mathfrak{S}(\mu)\mathfrak{Z}(\rho), \quad \mathsf{v} = \mathfrak{S}(\rho)\mathfrak{Z}(\mu).$$

$$U = \frac{\mathfrak{S}(\mu)}{\mathfrak{Z}(\mu)}, \quad \mathsf{v} = \frac{\mathfrak{S}(\rho)}{\mathfrak{Z}(\rho)}, \quad \mathfrak{Z}(\mu), \mathfrak{Z}(\rho) \neq 0.$$

$$U = \frac{\mathfrak{S}(\rho)}{\mathfrak{S}(\mu)}, \quad \mathsf{v} = \frac{\mathfrak{Z}(\rho)}{\mathfrak{Z}(\mu)}, \quad \mathfrak{Z}(\mu), \mathfrak{Z}(\rho) \neq 0. \quad \square$$

V. CONCLUSIONS

Numerous generalizations, extensions, and versions of the Grüss inequality and the Chebyshev inequality have been developed thanks to their widespread study. In this study, numerous extensions of the Grüss inequality and the Chebyshev-type inequality are shown using a generalized integral operator, namely the non-conformable operator. Because of the unique nature of the fractional operator and certain inequalities used in the proofs, these results provide fresh perspectives on the Grüss inequality. Researchers may improve their findings by using various kinds of fractional integral operators in future studies.

REFERENCES:-

1. El Shaed, M.A. Fractional Calculus Model of Semilunar Heart Valve Vibrations. In Proceedings of the International Mathematica Symposium, London, UK, 10–13 June 2003.

2. Hoan, L.V.C.; Akinlar, M.A.; Inc, M.; Gomez-Aguilar, J.F.; Chu, Y.M.; Almohsen, B. A new fractional-order compartmental disease model. Alex. Eng. J. 2020, 59, 3187–3196.

3. Gul, N.; Bilal, R.; Algehyne, E.A.; Alshehri, M.G.; Khan, M.A.; Chu, Y.M.; Islam, S. The dynamics of fractional order Hepatitis B virus model with asymptomatic carriers. Alex. Eng. J.

4. Kulish, V.V.; Lage, J.L. Application of fractional calculus to fluid mechanics. J. Fluids Eng. 2002,

5. Sahoo, S.K.; Tariq, M.; Ahmad, H.; Aly, A.A.; Felemban, B.F.; Thounthong, P. Some Hermite-Hadamard-type fractional integral inequalities involving twice-differentiable mappings. Symmetry 2021,

6. Rahman, G.; Abdeljawad, T.; Jarad, F.; Khan, A.; Nisar, K.S. Certain inequalities via generalized proportional Hadamard fractional integral operators. Adv. Diff. Eqs. 2019, 454, 1–10.