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IMPORTANCE AND APPLICATIONS OF SPECIAL FUNCTIONS IN MATHEMATICS

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ABSTRACT

A branch of mathematics of utmost importance to scientists and engineers concerned with real mathematical calculations is addressed in this study. The reader will find a systematic treatment here of the fundamental theory of the most important specific functions, as well as applications of theory to specific physics and engineering problems.

Keywords: - Mathematics, Function, Properties, Equation, Application.

I. INTRODUCTION

This research provides an introduction to the well-known classical special functions that play a role in mathematical physics, especially in major problems of boundary value. This branch of mathematics has a respectable history with great names, including Gauss, Euler, Fourier, Legendre, Bessel, and Riemann. All of them spent a lot of time on this topic. A good portion of their work was inspired by physics and the resulting differential equations. These activities culminated in the standard work of Whittaker and Watson about 70 years ago, A Course of Modern Analysis, which has had a great influence and is still important. As well as in applied fields such as electric current, fluid dynamics, heat conduction, wave equation, and quantum mechanics, special functions have extensive applications for more details in pure mathematics.

II. APPLICATION OF SPECIAL FUNTION

Special functions are mathematical functions that have specific properties or applications in various areas of mathematics, physics, engineering, and other scientific fields. They often arise in solving differential equations and have specific properties that make them useful in specific contexts. Here are some important applications of special functions:

Bessel Functions: Bessel functions are solutions to Bessel's differential equation and have applications in many areas, including wave propagation, heat conduction, electromagnetic theory, and signal processing. They are particularly useful in problems involving cylindrical symmetry, such as the diffraction of waves around circular obstacles.

Legendre Functions: Legendre functions are solutions to Legendre's differential equation and are widely used in classical mechanics, electrodynamics, and quantum mechanics. They play a crucial role in solving boundary value problems on spheres and other symmetric domains and appear in the expansion of functions in spherical harmonics.

Gamma Function: The gamma function is an extension of the factorial function to complex numbers. It has applications in many areas of mathematics and physics, such as probability theory, combinatorics, number theory, and quantum mechanics. It is also used to define other special functions, such as the beta function and the error function.

Hypergeometric Functions: Hypergeometric functions are solutions to hypergeometric differential equations and have diverse applications in mathematics, physics, and statistics. They are used in solving problems involving series expansions, orthogonal polynomials, conformal mapping, and the solution of linear differential equations.

Error Function: The error function, also known as the Gauss error function, is widely used in probability theory, statistics, and mathematical physics. It appears in the solution of diffusion equations, error analysis, and the evaluation of integrals involving normal distributions.

Lambert W Function: The Lambert W function is the inverse function of the equation y = xe^x and has applications in many areas, including physics, engineering, and economics. It arises in problems involving exponential growth and decay, population dynamics, and the solution of transcendental equations.

Zeta Function: The Riemann zeta function is a complex function that has applications in number theory, particularly in the study of prime numbers. It appears in the famous Riemann Hypothesis and has connections to other special functions, such as the gamma function and the Bernoulli numbers.

These are just a few examples of the wide range of special functions and their applications. Special functions are essential tools for mathematicians and scientists working on various problems, providing elegant and efficient solutions in many areas of study.

III. HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

Appel's Functions

In 1880 P. Appel (1855-1930) considered the product of two Gauss functions, viz.

$${}_2F_1(a,b;c;x)\quad {}_2F_1(a',b':c';y)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!}$$

This double series, in itself, yields nothing new, but if one or more of the three pairs of products

$$(a)_m (a')_n ,\ (b)_m (b')_n ,\ (c)_m (c')_n$$

be replaced by the corresponding expressions

$$(a)_{m+n},\ (b)_{m+n},\ (c)_{m+n},$$

we are led to five distinct possibilities of getting new functions. One such possibility, however, gives us the double series

$$\sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\ (b)_{m+n}}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}$$

which is simply the Gaussian series for

$${}_2F_1(a,b;c;x+y),$$

since it is easily verified thatt {cf, e.g., [85], p.4)}.

$$\sum_{m,n=0}^{\infty} f(m+n)\ \frac{x^m}{m!} \frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N)\ \frac{(x+y)^N}{N!},$$

or, more generally.

$$\sum_{m_1,\cdots\cdots m_n=0}^{\infty} f(m_1+\cdots\cdots+m_n)\ \frac{x_1^{m_1}}{m_1!} \cdots\cdots \frac{x_n^{m_n}}{m_n!}$$

$$= \sum_{m=0}^{\infty} f(m)\ \frac{(x_1+\cdots\cdots+x_n)^m}{m!}$$

The remaining four alternatives lead to the four Appell functions of two variables, which are explained further down in this paragraph:

$$F_1(a,b,b';c;x,y)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!},$$

$$\max\{|x|,|y|\} < 1;$$

$$F_2(a,b,b';c,c';x,y)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m (b')_n}{(c)_m\,(c')_n} \frac{x^m}{m!} \frac{y^n}{n!},$$

$$|x|+|y| < 1;$$

$$F_3(a,a',b,b';c;x,y)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_m\,(a')_n (b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!},$$

$$\max\{|x|,|y|\} < 1;$$

$$F_4(a,b;c,c';x,y)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}}{(c)_m\,(c')_n} \frac{x^m}{m!} \frac{y^n}{n!},$$

$$\sqrt{|x|}\,\sqrt{|y|} < 1.$$

The functions F_1, F_2, F_3 and F_4 given above are all generalization of the Gauss

hypergeometri c function ${}_2F_1$ given by (1.2.20).

Here, as usual, the denominator parameters c and c' are neither zero nor a negative integer. The standard work on the theory of Appel series is the monograph by Appel and Kempe' de Ferrite . See Erdely et al [19] for a review of the subsequent work on the subject; see also Bailey, Slater and Exton

Humbert's Functions

In the year 1920, Humbert [30] compiled a list of the seven confluent forms of the

four Appell's functions and designated them with the symbols

$\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2.$

Here we list two Humbert functions which are used in our subsequent work.

$$\Phi_3(\beta;\gamma;x,y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!},$$

$$|x| < \infty, |y| < \infty;$$

$$\Psi_2(\alpha;\gamma,\gamma';x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_m (\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!},$$

$$|x| < \infty, |y| < \infty.$$

Horn's Functions

The efforts of Appell were continued by Horn (1867-1946), who in the year 1931, defined ten hypergeometric functions of two variables and denoted them by

$G_1, G_2, G_3, H_1, \cdots\cdots, H_7.$ MT He thus completed the set of all possible complete

hypergeometric functions of two variables see {[94], p.56-57} and Erdely et al {[19],p.224-228}.

One of them are given below

$$H_3(\alpha,\beta;\gamma;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!},$$

$$|x| < r, |y| < s, r + (s - \tfrac{1}{2})^2 = \frac{1}{4}.$$

$$H_3(\alpha,\beta;\gamma;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!},$$

$$|x| < r, |y| < s, r + (s - \tfrac{1}{2})^2 = \frac{1}{4}.$$

An interesting result involving Appel's F2 and Horn's H3 functions was given by

Srivastava {[82], p.681 (2.2)}

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n w^n}{n!} F_2\left[\alpha,-n,-n;\mu,\lambda;\, x,y\right] z^n = (1-z)^{\alpha-\lambda}\, \theta^{-\alpha}$$

$$\times H_3\left[\alpha,\mu-\lambda;\mu;\frac{xyz}{\theta^2},\frac{xz}{\theta}\right],$$

where $\theta = 1 - z + xz + yz.$

Kampe de Feriet Function Appell's four double hypergeometric functions

F_1, F_2, F_3 and F_4, were unified and generalized by Kampe de Ferrite {[31], p.401404} (see also [4], p. 150 (29)).

We recall the definition of general double hypergeometric function of Kampe de Feriet in the slightly modified notation of Srivastava and Panda

$$F^{A:B;D}_{E:G;H}\left[\begin{matrix}(a_A);(b_B);(d_D);\\(e_E);(g_G);(h_H);\end{matrix} x,y\right] = \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{m+n}\,[(b_B)]_m\,[(d_D)]_n}{[(e_E)]_{m+n}\,[(g_G)]_m\,[(h_H)]_n}\, \frac{x^m}{m!}\, \frac{y^n}{n!},$$

where, for convergence,

$A + B < E + G + 1, \quad A + D < E + H + 1, |x| < \infty, |y| < \infty,$ *or*

$A + B = E + G + 1, \quad A + D = E + H + 1,$ *and*

$$\begin{cases} |x|^{1/(A-E)}\,|y|^{1/(A-E)} < 1, & \text{if } A > E,\\ \max\ \{|x|,|y|\} < 1, & \text{if } A \le E.\end{cases}$$

IV. HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

Lauricella Function of n Variables

Lauricella [41] generalized the Appell double hypergeometric functions $F_1,\cdots\cdots,F_4$

(cf. e.g., [19], p.224) to functions of n variables. Two of Lauricella functions, viz.

$F_A^{(n)}$ and $F_D^{(n)}$ are defined by

$$F_A^{(n)}(a, b_1, \cdots\cdots, b_n; c_1, \cdots\cdots, c_n; x_1, \cdots\cdots, x_n)$$

$$= \sum_{m_1, \cdots\cdots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots\cdots + m_n} (b_1)_{m_1} \cdots\cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots\cdots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots\cdots \frac{x_n^{m_n}}{m_n!},$$

$$|x_1| + \cdots\cdots + |x_n| < 1;$$

$$F_D^{(n)}(a, b_1, \cdots\cdots, b_n; c; x_1, \cdots\cdots, x_n)$$

$$= \sum_{m_1, \cdots\cdots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots\cdots + m_n} (b_1)_{m_1} \cdots\cdots (b_n)_{m_n}}{(c)_{m_1 + \cdots\cdots + m_n}} \frac{x_1^{m_1}}{m_1!} \cdots\cdots \frac{x_n^{m_n}}{m_n!},$$

$$\max\{|x_1|, \cdots\cdots, |x_n|\} < 1.$$

Clearly, we have,

$$F_A^{(2)} = F_2 \text{ and } F_D^{(2)} = F_1 .$$

Lauricella [40] outlined a few of the fundamental characteristics of these functions as well. Appell and Kampe de Feriet both provide a synopsis of the work that Lauricella has done.

Confluent Forms of Lauricella Functions

Two important confluent hypergeometric functions of n variables are the functions

Of $\Phi_2^{(n)}$ and $\Psi_2^{(n)}$ (See e.g. [94] p.62}.

Here we need $\Psi_2^{(n)}$ only, which defined by

$$\Psi_2^{(n)}[a; c_1, \cdots\cdots, c_n; x_1, \cdots\cdots, x_n]$$

$$= \sum_{m_1, \cdots\cdots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots\cdots + m_n}}{(c_1)_{m_1} \cdots\cdots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots\cdots \frac{x_n^{m_n}}{m_n!},$$

clearly, we have

$\Psi_2^{(2)} = \Psi_2$, where Ψ_2 is Humbert confluent hypergeometric function of two

variables defined by (1.3.9).

Generalized Lauricella Functions of Several Variables

A further generalization of the Kampe de Feriet Function of two variables $F_{E:G:H}^{A:B;D}$

andLauricella functions of several variables $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and is due to

Srivastava and Daoist

The generalized Lauricella functions of n variable is defined as follows:

$$F\begin{matrix} A:B';\cdots\cdots;B^{(n)} \\ C:D';\cdots\cdots;D^{(n)} \end{matrix}\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \equiv F\begin{matrix} A:B';\cdots\cdots;B^{(n)} \\ C:D';\cdots\cdots;D^{(n)} \end{matrix}\left(\begin{matrix} [(a):(\theta'),\cdots\cdots,\theta^{(n)}]: \\ [(c):\psi',\cdots\cdots,\psi^{(n)}]: \end{matrix}\right.$$

$$\left.\begin{matrix} [(b'):(\phi')];\cdots\cdots;[(b^{(n)}):(\phi^{(n)})]; \\ [(d'):\delta'];\cdots\cdots;[(d^{(n)}):(\delta^{(n)})]; \end{matrix} z_1,\cdots\cdots,z_n\right)$$

$$= \sum_{m_1,\cdots\cdots m_n=0}^{\infty} \Omega(m_1,\cdots\cdots,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\cdots\frac{z_n^{m_n}}{m_n!}. \qquad \cdots\cdots$$

where, for convenience,

$$\Omega(m_1,\cdots\cdots,m_n)$$

$$= \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j'+\cdots\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b_j')_{m_1\phi_j'}\cdots\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j'+\cdots\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d_j')_{m_1\delta_j'}\cdots\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

the coefficients

$$\begin{cases} \theta_j^{(k)}, j=1,\cdots\cdots,A \;\; ;\phi_j^{(k)}, j=1,\cdots\cdots,B^{(k)} \;;\psi_j^{(k)}, j=1,\cdots\cdots,C\,; \\ \delta_j^{(k)}, j=1,\cdots\cdots,D^{(k)} \;\; ; \quad \forall k\in\{1,\cdots\cdots,n\} \end{cases}$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\cdots\cdots,a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters

$b_j^{(k)}$, $j=1,\cdots\cdots,B^{(k)}$; $\forall k\in\{1,\cdots\cdots,n\}$,

with similar interpretations for and $(d^{(k)})$, $k=1,\cdots\cdots,n$ etcetera.

A detailed discussion of the conditions of convergence of the multiple series (1.4.4) is given in Srivastava and Doust [89], if the positive constants $\theta's$, $\psi's$, $\phi's$ and $\delta's$ are all chosen as unity then (1.4.4) reduces to the generalized Kampe de Feriet function given by Karlsson in its more general form.

$$F\begin{matrix}A:B';\cdots\cdots;B^{(n)}\\C:D';\cdots\cdots;D^{(n)}\end{matrix}[z_1,\cdots\cdots,z_n]$$

$$= F\begin{matrix}A:B';\cdots\cdots;B^{(n)}\\C:D';\cdots\cdots;D^{(n)}\end{matrix}\left[\begin{matrix}(a):(b');\cdots\cdots;(b^{(n)})\ ;\\(c):(d');\cdots\cdots;(d^{(n)})\ ;\end{matrix}z_1,\cdots\cdots,z_n\right]$$

$$= \sum_{m_1,\cdots\cdots,m_n=0}^{\infty}\frac{((a))_{m_1+\cdots\cdots+m_n}((b'))_{m_1}\cdots\cdots((b^{(n)}))_{m_n}\ z_1^{m_1}\cdots\cdots z_n^{m_n}}{((c))_{m_1+\cdots\cdots+m_n}((d'))_{m_1}\cdots\cdots((d^{(n)}))_{m_n}\ m_1!\cdots\cdots m_n!}.$$

clearly, we have

$$F\begin{matrix}1:1;\cdots\cdots;1\\0:1;\cdots\cdots;1\end{matrix}=F_A^{(n)}\ ,\ F\begin{matrix}0:2;\cdots\cdots;2\\1:0;\cdots\cdots;0\end{matrix}=F_B^{(n)}\ ,$$

$$F\begin{matrix}2:0;\cdots\cdots;0\\1:1;\cdots\cdots;1\end{matrix}=F_C^{(n)}\ and\ F\begin{matrix}1:1;\cdots\cdots;1\\1:0;\cdots\cdots;0\end{matrix}=F_D^{(n)}\ .$$

The Triple Hypergeometric Functions of Lauricella – Saran

Lauricella {[41], p. 114} introduced fourteen complete hypergeometric functions of three variables and of the second order. He denoted his triple hypergeometric functions by the symbols

$\boldsymbol{F_1, F_2, F_3, \ldots\ldots, F_{14}}$ of which $\boldsymbol{F_1}$, $\boldsymbol{F_2}$, $\boldsymbol{F_3}$, and $\boldsymbol{F_9}$ corrsepond,

respectively, to the three-variable Lauricella functions $F_A^{(3)}, F_B^{(3)}, F_C^{(3)}$ and $F_D^{(3)}$

with n=3. The remaining ten functions, $F_3, F_4, F_6, F_7, F_8, F_{10}, \ldots\ldots, F_{14}$ of

Lauricella's set apparently fell into oblivion [except that there is an isolated

appearance of the triple hypergeometric function F_8 in a paper by Mayr {[46], p.265}. Saran [76] initiated a systematic study of these ten triple hypergeometric functions of Lauricella's set. We give below the definitions of four of these functions using Saran's

notations and and also indicating Lauricella's notations:

$$F_G(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$

$$=\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma_1)_m,(\gamma_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$|x|<r,|y|<s,|z|<t,\quad r+s=1=r+t;$$

$$F_K(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z)$$

$$=\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$|x|<r,|y|<s,|z|<t,\quad (1-r)(1-s)=t;$$

$$F_N(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$

$$=\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1)_m(\alpha_2)_n(\alpha_3)_p(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$|x|<r,|y|<s,|z|<t,\quad (1-r)s+(1-s)t=0;$$

$$F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_1,\gamma_1;x,y,z)$$

$$=\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma_1)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$|x|<r,|y|<s,|z|<t,\quad r+s=rs,\quad s=t.$$

The General Triple Hypergeometric Series $F^{(3)}[x,y,z]$

A unification of Lauricella's fourteen-hypergeometric functions $F_1, \cdots\cdots, F_{14}$ of three variables {[41], p.ll4} and Srivastava's three additional funcdons H_A, H_B, H_C [81], was introduced by Srivastava (see. e.g. {[80], p.428} and {[94],

p.69}) in the forni of triple hypergeometric series $F^{(3)}\ [x, y, z]$ defined as

$$F^{(3)}[x,y,z] = F^{(3)}\begin{bmatrix} (a)::(b);(b');(b''):(c);(c');(c''); & \\ & x,y,z \\ (e)::(g);(g');(g''):(h);(h');(h''); & \end{bmatrix}$$

$$= \sum_{m,n,p=0}^{\infty} \frac{((a))_{m+n+p}\,((b))_{m+n}\,((b'))_{n+p}\,((b''))_{p+m}\,((c))_m\,((c'))_n\,((c''))_p\, x^m y^n z^p}{((e))_{m+n+p}\,((g))_{m+n}\,((g'))_{n+p}\,((g''))_{p+m}\,((h))_m\,((h'))_n\,((h''))_p\, m!n!p!}$$

where {a) and ((a)) will mean the sequence of a parameters $a_1, \cdots\cdots, a_A$ and the product

$\prod_{j=1}^{A}(a_j)$ respectively.

The triple hypergeometric series converges absolutely when

$$\begin{cases} 1+E+G+G''+H-A-B-B''-C\geq 0, \\ 1+E+G+G'+H'-A-B-B'-C'\geq 0, \\ 1+E+G'+G''+H''-A-B'-B''-C''\geq 0, \end{cases}$$

where the equality holds true for the values that have been appropriately limited

$|x|, |y|$ *and* $|z|$.

The Multiple Hypergeometric Functions ${}^{(k)}_{(1)}E^{(n)}_{D}$ and ${}^{(k)}_{(2)}E^{(n)}_{D}$

Exton {[23], p. 89 (3.4.1), (3.4.2)} considered the two multiple hypergeometric functions which follow as a generalization of certain of quadrable functions.

The functions are defined as follows:

$${}^{(k)}_{(1)}E_D^{(n)}\left[a,b_1,\cdots\cdots,b_n;c,c';x_1,\cdots\cdots,x_n\right]$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots\cdots+m_n}(b_1)_{m_1}\cdots\cdots(b_n)_{m_n}x_1^{m_1}\cdots\cdots x_n^{m_n}}{(c)_{m_1+\cdots\cdots+m_k}(c')_{m_{k+1}+\cdots\cdots+m_n}m_1!\cdots\cdots m_n!}.$$

And

$${}^{(k)}_{(2)}E_D^{(n)}\left[a,a',b_1,\cdots\cdots,b_n;c;x_1,\cdots\cdots,x_n\right]$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots\cdots+m_k}(a')_{m_{k+1}+\cdots\cdots+m_n}(b_1)_{m_1}\cdots\cdots(b_n)_{m_n}x_1^{m_1}\cdots\cdots x_n^{m_n}}{(c)_{m_1+\cdots\cdots+m_k}m_1!\cdots\cdots m_n!}$$

……

The convergence of the functions (1.4.13) and (1.4.14) are given in

We not the following special cases:

$${}^{(1)}_{(1)}E_D^{(2)}\left[a,b_1,b_2;c_1,c_2;x,y\right]$$

$$=F_2\left[a,b_1,b_2;c_1,c_2;x,y\right]$$

where F2 is Appell function of two variables defined by

$${}^{(1)}_{(1)}E_D^{(3)}\left[a,b_1,b_2,b_3;c,c';x,y,z\right]$$

$$=F_G\left(a,a,a,b_1,b_2,b_3;c,c',c';x,y,z\right)$$

whereFg is Lauricella-Saran function of three variables defined by

$${}^{(3)}_{(1)}E_D^{(4)}\left[a,b_1,b_2,b_3,b_4;c,c';w,x,y,z\right].$$

$$=K_{11}\left(a,a,a,a,b_1,b_2,b_3,b_4;c,c,c,c';w,x,y,z\right).$$

where K_{11} is Exton's quadriplehypergeometric functions

$$K_{11}\left(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;w,x,y,z\right)$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q w^m, x^n y^p z^q}{(c)_{m+n+p}(d)_q m!n!p!q!}.$$

$${}^{(1)}_{(2)}E_D^{(3)}\left[a,a',b_1,b_2,b_3;c;x,y,z\right]$$

$$=F_S\left(a,a',a',b_1,b_2,b_3;c,c,c;x,y,z\right)$$

where F_s is the Saran's function of three variables defined by

$${}^{(3)}_{(2)}E_D^{(4)}\left[a,a',b_1,b_2,b_3,b_4;c;w,x,y,z\right].$$

$$=K_{15}\left(a,a,a,a',b_1,b_2,b_3,b_4;c,c,c,c;w,x,y,z\right).$$

where K_{15} is Exton's quadriplehypergeometric functions

$$K_{15}\left(a,a,a,b_5,b_1,b_2,b_3,b_4;c,c,c,c;w,x,y,z\right)$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p}(b_5)_q(b_1)_m(b_2)_n(b_3)_p(b_4)_q w^m x^n y^p z^q}{(c)_{m+n+p+q} m!n!p!q!}.$$

V. CONCLUSION

In this study, we covered the fundamentals of using generating functions and special functions to address linear recurrence and combinatorial issues. Yet, there are undoubtedly a great deal more facets to the topic that are not covered here. Those curious about generating functions are encouraged to read on for a comprehensive overview of the topic.

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