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Applications of Higher Order Partial Differential Equations: An Analytical Solution

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Abstract: Separation of variables has long been acknowledged as one of the most effective methods for resolving linear partial differential equations (PDEs). In this research, an analytical solution to higher order homogeneous partial differential equations (PDEs) within a rectangular domain with specified boundary conditions (BCs) is proposed. Initially, the partial differential equation (PDE) is reduced to an ordinary differential equation (ODE) by means of variable separation and integral components. The analytical answer is obtained by using a power series expansion of the unknown function after symbolic manipulations. This paper presents a special instance of variable separation, where the PDE on one variable is solved by removing the other variable. It takes less work to implement the suggested closed form solution than the alternative numerical solutions, which are provided here. The efficiency of the developed approach demonstrates the capacity to solve higher order linear PDEs analytically while overcoming the complexity of mixed derivatives and boundary conditions.

Keywords: Power Series, PDE, and variables.

1. INTRODUCTION

In addition to empirical methods, various implementations of higher order linear PDEs have been used in recent decades for the representation of solid surfaces in shape design for engineering manufacturing. PDEs are used for the manipulation and representation of surface/solid models in computer-aided geometric design. In addition to their significance in the medical field for human tissue visualization and surgical simulation, they also had vital importance in engineering for analysis and simulation. Many engineering applications' exact linear PDE solutions are still a mystery to mathematicians and specialized publications. Early in the 1950s, M. H. Martin (1953) noted a unique instance of a two-term equation that could be obtained by applying variable separation to Laplace's equation. His notation marked a paradigm shift for conventional reduction techniques by providing fresh perspectives on what is now recognized as functional separation of variables.

Karimov and Pirnafasov (2017) study the higher order multi-term PDE with fractional derivative in time. To decrease the fractional order PDE to the integer order, they employed variable separation. Rakhmelevich (2016) simplified a multi-dimensional partial differential equation using a linear differential operator and used variable separation to solve it. Everitt and Johansson (2019) solved the Dirichlet issue for the biharmonic PDE on a confined region by using quasi-separation of variables. A fourth-order parabolic equation's nonlocal problem was examined by Berdyshev and Kadirkulov (2016) using the Dzhrbashyan-Nersesyan fractional differential operator. They used separation of variables to prove the existence and uniqueness of the solution of the problem. Publishers' attempts on the separation of variables of higher order PDE have already been previously done using different methods such as Fourier series and Galerkin formulation.

Highly accurate analytical solutions for many applications such as anisotropic and orthotropic rectangular plates are introduced by several researchers. In the present paper, a semi analytical technique is proposed relying on applying circumvent between generalized separation of variables and integral factors. The derived technique leads to exact closed form solution for higher order PDEs incorporating mixed derivatives. The present method is applied to higher order linear PDEs with constant coefficients.The researched PDEs and boundary conditions are used to build the shape function. Conversely, a polynomial of power series is used to express the unknown separation function. Under the suggested boundary conditions BCs, the developed technique is used to produce the closed form solution of composite plate vibration.

2. PARTIAL DIFFERENTIALEQUATION

The general higher order homogenous PDE is:

$$f\left(x,y,w,\frac{\partial^n w}{\partial x^n},\frac{\partial^n w}{\partial x^r\,\partial y^{n-r}},\dots\dots,\frac{\partial^n w}{\partial y^n},\frac{\partial^{n-1} w}{\partial x^{n-1}},\frac{\partial^{n-1} w}{\partial x^r\,\partial y^{n-r-1}},\dots\dots,\frac{\partial^{n-1} w}{\partial y^{n-1}},\frac{\partial^{n-2} w}{\partial x^{n-2}},\frac{\partial^{n-2} w}{\partial x^r\,\partial y^{n-r-2}},\dots,\frac{\partial^{n-2} w}{\partial y^{n-2}},\dots\dots,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right)=0; \qquad r=1,2,3,\dots\dots\dots,n-1 \quad \dots.(1)$$

Someplace,

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$w = w(x, y)$is the unidentified function of the reliant on variables x, y.

Let us reflect the dimensionless PDE finished a rectangular $r$district$(a \times b) \in \Re$ in the short procedure:

$$(Lw)(\zeta, \eta) = 0 \dots \dots \dots . . (2)$$

where w=w(μ,η) is the dimensionless unknown function and L is the linear partial differential operator.

$$\zeta = \frac{x}{a} \text{ and } \eta = \frac{x}{b} \dots \dots . (3)$$

Variable separation presupposes a general solution of the following kind:

$$w(\zeta, \eta) = \sum_{m=1}^{\mathrm{M}} g_m(\zeta) f_m(\eta) \dots \dots \dots \dots . . (4)$$

Where $g_m(\zeta)$ and $f_m(\eta)$are functions that, at the boundaries (η=0,1) and (ζ=0,1), respectively, meet the boundary requirements of the rectangular region. Equation (2) is reduced to the ordinary differential equation (ODE) by using equation (4):

$$\sum_{m=1}^{M} \sum_{n=0}^{N} C_{mn} f^{(n)}(\eta) = 0 \dots \dots \dots (4)$$

whereC_{mn} is an integrated constant based on the known function $g_m(\zeta)$ and the function $f^{(n)}(\eta)$is an unknown function differentiated with order n. This allows for:

$$C_{mn} = \int_0^1 g_m g_m^{(n)} d\zeta \dots \dots \dots (5)$$

The simplified ODE (5)'s series solution is:

$$w(\zeta, \eta) = \sum_{m=1}^{\mathrm{M}} g_m(\zeta) \left[f_m(0) + \sum_{k=1}^{K} f_m^{(k)}(0) \frac{\eta^k}{k!} \right] \dots \dots . . (6)$$

$$f_m^{(k)}(0) = \frac{d^k f}{d\eta^k} \quad \text{at} \quad \eta = 0; k = 1,2,3, \dots \dots . . . K \dots \dots \dots (7)$$

Thus:

$$w(\zeta, \eta) = \sum_{m=1}^{\mathrm{M}} g_m(\zeta) \Big\{ f_m(0) + f_m'(0)\eta + f_m''(0) \frac{\eta^2}{2!} \dots \dots \dots \dots . . (8)$$

$$+ f_m'''(0) \frac{\eta^3}{3!} + \cdots \dots + f_m^{(n-1)}(0) \frac{\eta^{n-1}}{(n-1)!} + f_m^{(n)}(0) \frac{\eta^n}{n!} + E_0 \Big\} \dots \dots \dots (9)$$

$$E_0 = \sum_{k=n}^{K} f_m^{(k)}(0) \frac{\eta^k}{k!} \dots \dots \dots \dots . (10)$$

3. SEPARATION OF VARIABLES

Under boundary and beginning conditions, the shape function of linear partial differential equations is created in only one variable by applying separation of variables.

$$w(\zeta,\eta) = g(\zeta)f(\eta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(11)$$

Separating variables is simple in situations like the heat equation, Laplace equation, Helmholtz equation, and wave equation that don't contain mixed derivatives. Even if it is difficult to isolate the PDE in situations involving mixed derivatives, such the biharmonic equation, Eq. (11) can still be used. Under boundary conditions, two single variable ODE systems are formed according to separation. Shape function is defined directly when it satisfies the suggested boundary conditions. The dimensionless biharmonic equation, for instance, is taken into consideration:

$$\nabla^2 w = 0 \quad \text{Where} \quad \nabla = \frac{\partial^2 w}{\partial \zeta^2} + \tau^2 \frac{\partial^2 w}{\partial \eta^2} \ldots\ldots\ldots(12)$$

$$\tau = \frac{a}{b}$$

In subsection (11) of (12), it is given:

$$\frac{g^{(4)}(\zeta)}{g(\zeta)} + 2\tau^2 \frac{g''(\zeta)}{g(x)} \frac{f''(\eta)}{f(\eta)} + \tau^4 \frac{f^{(4)}(\eta)}{f(\eta)} = 0 \ldots\ldots\ldots.(13)$$

$$G(\zeta) + 2\tau^2 E(\zeta)H(\eta) + \tau^4 F(\eta) = 0 \ldots\ldots\ldots..(14)$$

By differentiating Eq. (14) with respect to $\zeta$and η, the first and last terms are eliminated, yielding:

$$E'(\zeta)H'(\eta) = 0\ldots\ldots(15)$$

Thus,

$$-G(\zeta) = 2\tau^2 F(\zeta)H(\eta) + \tau^4 F(\eta) \quad \text{or} \ -\tau^4 F(\eta) = G(\zeta) + 2\tau^2 F(\zeta)H(\eta) \text{ is constant.}$$

Consequently, two situations are produced:

$$f^{(4)}(\eta) + 2\mu_1\tau^2 f''(\eta) + \tau^4\omega_1 f(\eta) = 0 \ldots\ldots(16)$$

$$g^{(4)}(\zeta) + 2\mu_2\tau^2 g''(\zeta) + \tau^4\omega_2 g(\zeta) = 0 \ldots\ldots\ldots\ldots.(17)$$

$$\frac{g''(\zeta)}{g(\zeta)} = \mu_1, \frac{g^{(4)}(\zeta)}{g(\zeta)} = \omega_1 \ldots\ldots\ldots(18)$$

$$\frac{f''(\eta)}{f(\eta)} = \mu_2, \frac{f^{(4)}(\eta)}{f(\eta)} = \omega_2 \ldots\ldots..(19)$$

Thus, one can demonstratethat $\omega_1 = \mu_1^2$.

Correspondingly$\omega_2 = \mu_2^2$

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The shape function $g_m(\zeta)$, which has the following form, is the general solution to equation (18):

$$g_m(\zeta) = A_1 \sin(\alpha_m \zeta) + A_2 \cos(\alpha_m \zeta) + A_3 \sinh(\alpha_m \zeta + A_4 \cosh(\alpha_m \zeta) \ldots\ldots (20)$$

where α_m and A_1, A_2, A_3, A_4 are constant parameters that rely on τ, μ2, and boundary conditions at the support's edges. Numerous academics have published descriptions of generalised methods for variable separation in literature.

4. Numerical Results

Here, the vibration case of a square simply supported SSSS orthotropic plate is studied in order to realise the explicit and implicit solutions derived from equation (18) under given boundary conditions. The material characteristics and bending stiffness of an orthotropic carbon-epoxy plate are as follows: the plate thickness is 0.75 " mm.

$D11 = 409.08, D22 = 23.28, D12 = 6.52, D16 = D26 = 0$ and $D66 = 16.13$

In this instance, the power series' eigenvalues λ_{mn} are obtained, either directly or implicitly, for truncation number k=12. Three vibration modes are investigated, and the findings are as follows:

$m = 1$

The overteigen values $\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \ldots\ldots\ldots \lambda_{1n}$ when $m = 1$ are gotten from the explanation of the subsequent algebraic equation:

$$1.809214146(10^{19}) - 2.015964353(10^{16})\lambda_1^2 +$$
$$8.920238451(10^{12})\lambda_1^4 - 2.206726296(10^9)\lambda_1^6 \ldots\ldots\ldots (21)$$
$$+2.623401997(10^5)\lambda_1^8 = 0 \ldots\ldots. (22)$$

So, the first 3 values of λ_{1n} stand

$$\lambda_{11} = 44.71021071, \quad \ldots\ldots..(23)$$

$$\lambda_{12} = 56.24689473, \quad \ldots\ldots..(24)$$

$$\lambda_{13} = 49.63224921 \ldots\ldots\ldots.(25)$$

To illustrate the effectiveness of the present technique, the eigen function (mode shape) corresponding to an eigen value, say $\lambda_{11} = 44.71021071$, is expressed in explicit closed form:The eigen function (mode shape) corresponding to an eigen value, such as λ_{11}=40.71121061, is expressed in explicit closed form to demonstrate the efficacy of the current method:

$$w_{11}(\zeta, \eta) = \Big[1.0(10)^9\zeta + .5\zeta^2 - 1.636255768(10)^9\zeta^3 + .8003096258\zeta^4 + 8.134474111(10)^8\zeta^5$$
$$+ 0.1477452619\zeta^{10} - 2.342321687(10)^6\zeta^{11}\Big][\sin \pi\eta] = 0 \ldots\ldots\ldots.. (26)$$

In Fig. 1, this mode shape is shown.

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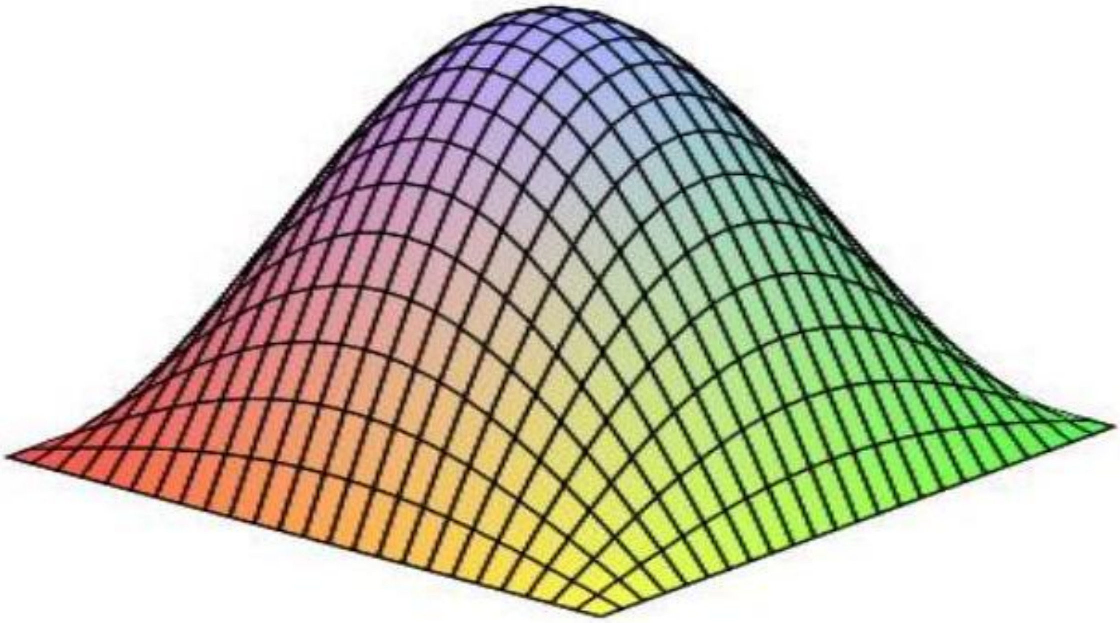

Fig. 1: Plate First Mode Shape

$\boldsymbol{m = 2}$

Correspondingly, the explicit eigen value $\lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{24}, \dots \dots ..\lambda_{2n}$ are articulated in:

$$2.077943337(10)^{23} - 2.669490059(10)^{19}\lambda_2{}^{2} + 1.378822461(10)^{15}\lambda_2{}^{4} - 3.108682006(10)^{10}\lambda_2{}^{6}) + 2.61340101(11)^{5}\lambda_2{}^{8} = 0 \dots \dots \dots \dots (27)$$

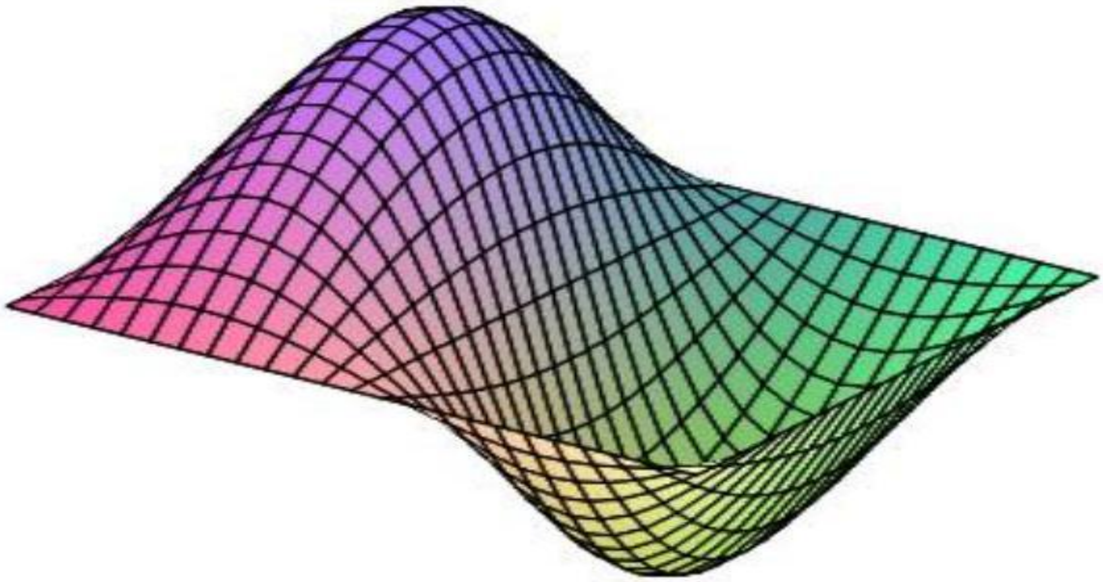

Fig. 2: Plate Second Mode Shape

Thus, the 1st three values of λ_{2n} stand

$$\lambda_{21} = 157.05534469,$$

$$\lambda_{22} = 169.0454913,$$

$$\lambda_{23} = 169.5320574$$

Additionally, the plate's mode shape corresponds to:

$\lambda_{21} = 170.05535379$isexemplified in Fig. 2. andcharacterizedthru the eigen function:

$$w_{21}(\zeta,\eta) = [1.0(10)^9\zeta + .5100010000\zeta^2 - 1.643483498(10)^9\zeta^3 + 3.2108599398$$
$$+8.133347156(10)^8\zeta^5 + 9.375249633\zeta^6 - 1.918624327(10)^8\zeta^7 + 14.40536273\zeta^8$$
$$+2.627345926(10)^7\zeta^9 + 13.95532167\zeta^{10} - 2.272221645(10)^6\zeta^{11}][\sin 2\pi\eta] = 0 \dots\dots (28)$$

$\boldsymbol{m = 3}$

The obviouseigen values $\lambda_{31}, \lambda_{32}, \lambda_{33}, \lambda_{34}, \dots \dots \dots \lambda_{3n}$ are gainedsince:

$$2.078941326(10)^{23} - 2.759290149(10)^{19}\lambda_3{}^2 + 1.378822472(10)^{15}\lambda_3{}^4 - 3.108683017(10)^{10}\lambda_3{}^6 \dots (29)$$

In this circumstance, the 1st three values of λ_{3n}remain

$$\lambda_{31} = 369.7592815$$

$$\lambda_{32} = 379.9458546,$$

$$\lambda_{33} = 379.4063672$$

Also, the Eigen function conforming to $\lambda_{31} = 374.7693805$ is characterized by the subsequent equation and showedexplicitly in Fig. 3.

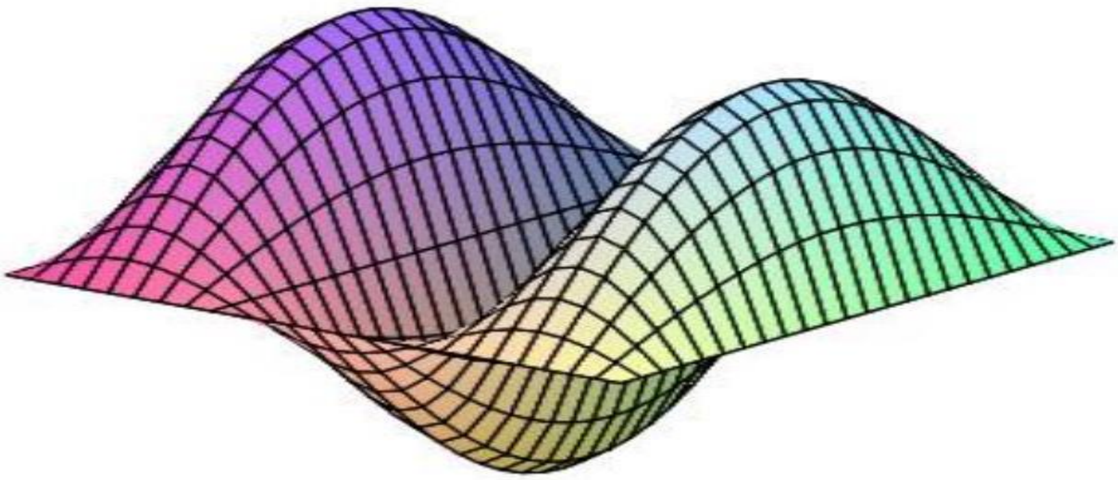

Fig. 3: Plate Third Mode Shape

$$w_{31}(x,y) = [1.0(10)^9\zeta + .5^*\zeta^2 - 1.642311542(10)^9\zeta^3 + 7.203425912\zeta^4$$
$$+143.6524873\zeta^8 + 2.573851628(10)^7\zeta^9 + 291.7429516\zeta^{10}$$
$$-2.500954608(10)^6\zeta^{11}][\sin 3\pi\eta] = 0 \dots\dots (29)$$

5. CONCLUSUN

A revised iteration of the variable separation approach is showcased. Mathematical manipulation gets more challenging when one considers mixed derivatives in the governing PDEs and direct substitutions of boundary conditions. As a result, there are not many publications of this kind in literature. The exact method that is provided spares the superfluous work required to achieve the other numerical solutions. The presented methodology converts a set of higher-order linear PDEs with mixed derivatives into a manageable number of ODEs. Under specific boundary conditions, the application of the current technique yields straightforward, straightforward, and extremely accurate closed form solutions for engineering problems controlled by higher order linear PDE involving mixed derivatives.The technique's validity and reliability are assessed by using it to analyse three types of orthotropic plate vibration. The solutions are presented by the application in a straightforward, accurate analytical manner.

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