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Application of the Partial Differential Equations using the D' Alembert's Formula

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Abstract: In this paper, we consider several situations stemming from the applications, and the mathematical modeling of which involves partial differential equation problems. Our primary focus in these research projects is on the good qualities and consequences of a specific partial differential equation's solution. The homogeneous one-dimensional wave equation in particular piques our interest in the mathematical modelling of the consistency and well-posedness of the solution or solutions to certain PDEs. A function u = u(x, y, z, t) will be used to measure different physical quantities. We examine the homogeneous one-dimensional wave equation via the lens of mathematical modelling of partial differential equations. Specifically, we investigate the solution's well-posedness and consistency (Guo and Zhang, 2007). The method of change of variable is to be used to derive the d' Alembert's general solution, which will ultimately lead us to the d' Alembert's formula for the wave equation solution. Though the classical theory of partial differential equations deals almost completely with the well-posed, ill posed problems can be mathematically and scientifically interesting. After that, we analyzed the results using the answer we had acquired, displayed the behavior of our results in a table, and came to the conclusion that the idea of a well-posed issue is crucial in applied mathematics.

Keyword: Partial Derivatives; PDE, Modelling

1. Introduction

In general, it may be impossible or at the very least difficult to find the exact solution to partial differential equation problems. Partial differential equations were initially studied as a means of examining physical science models. Physical rules like momentum, conservation laws, balancing forces (Newton's law), and others are the usual source of PDEs (Strauss, 2008). This paper derives the string's equation of motion, which takes the form of a second-order partial differential equation, under specific assumptions. The one-dimensional wave equation, or governing partial differential equation, depicts the transverse vibration of an elastic string (King and Billingham, 2000). The change of variable method has been used to obtain the analytical answer.One of the most important mathematical puzzles of the middle of the eighteenth century was the solution to the wave equation. D'Alembert derived and investigated the wave equation for the first time in 1746. He presented the equation for a one-dimensional wave.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \text{.......}(1)$$

We next generalized the wave equation to two and three dimensions, i.e.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta u(x,t) \text{.......}(2)$$

Everyplace

$$\Delta = \sum_{i-1}^{3} \frac{\partial^2}{\partial x_i^2} \text{.....}(3)$$

In a number of works, the wave equation's solution was found in a variety of ways (Benzoni Gavage and Serre, 2007). Models of the most fundamental theories underpinning physics and engineering are frequently created using partial differential equations (Lawrence, 2010). There is no one theorem that is essential to the subject, in contrast to the theory of ordinary differential equations, which depends on the fundamental existence and uniqueness theorem. Rather, distinct theories are used to every major class of partial differential equations that frequently occur. It is important to note that differential equations of either first or second order, with the latter being by far the more common, are the majority when they arise in applications, science, engineering, and mathematics itself. A partial derivative of the independent variable, which is an unknown function in multiple variables, is found in a PDE x, y, t,

$$\frac{\partial u}{\partial x} = u_x, \frac{\partial u}{\partial y} = u_y \& \frac{\partial u}{\partial t} = u_t \text{.......}(4)$$

The universal first order PDE for u(x, t) can be expressed as

$$F[x,t,u(x,t),u(x,t),u(x,t)] = F(x,t,u,u_x,u_t) = 0 \ldots\ldots..(5)$$

While PDEs in multiple independent variables can be studied, PDEs in two independent variables will be the main focus of this research project.

2. Statement of the problem

Our primary focus in these research projects is on the good qualities and consequences of a specific partial differential equation's solution. The homogeneous one-dimensional wave equation in particular piques our interest in the mathematical modelling of the consistency and well-posedness of the solution or solutions to certain PDEs. A function u = u(x, y, z, t) will be used to measure different physical quantities. This function may depend on all spatial variables and time or only on a subset of them (Guo 2009). The shortened notation that follows will be used to represent the partial derivatives of u:

$$u_x = \frac{\partial u}{\partial y}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{xy} = \frac{\partial^2 u}{\partial x \, \partial y}, u_{xt} = \frac{\partial^2 u}{\partial x \, \partial t}, u_t = \frac{\partial u}{\partial t} \dots \dots (6)$$

3. Objectives

- To find a solution, use the method of change of variable to assess the homogeneous one-dimensional wave equation's solution for consistency and well-posedness.
- Application of the Partial Differential Equations using the D' Alembert's formula

4. METHODS

We will look at a specific kind of problem related to partial differential equations that are hyperbolically linear. This issue will be discussed in relation to the homogeneous one-dimensional wave equation of the type

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \dots\dots\dots (7)$$

where C is a constant and the independent variables are x and t.

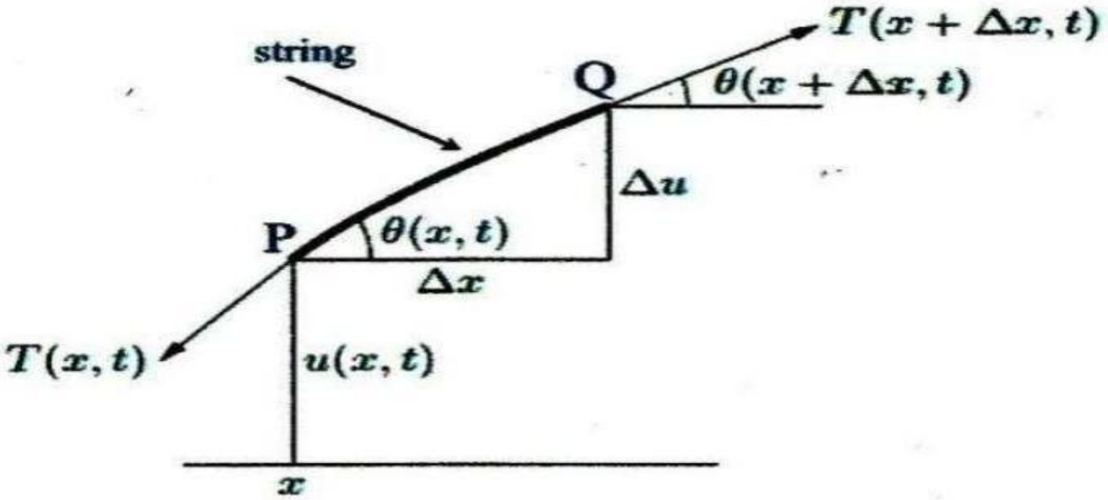
This equation is the model for a class of hyperbolic differential equations and is known as the homogeneous one-dimensional wave equation. In practical applications, hyperbolic equations are used to simulate various types of waves, including gravitational, elastic, electromagnetic, and acoustic waves. When compared to parabolic and elliptic PDEs, the qualitative characteristics of hyperbolic PDEs are significantly different.Undoubtedly, one of the most significant classical equations in mathematical physics is the wave equation. We examine the homogeneous one-dimensional wave equation via the lens of mathematical modelling of partial differential equations. Specifically, we investigate the solution's well-posedness and consistency (Guo and Zhang, 2007). The method of change of variable is to be used to derive the d'Alembert's general solution, which will ultimately lead us to the d'Alembert's formula for the wave equation solution.

5. **Mathematical Formulation**

There are numerous physical applications for the wave equation, ranging from sound waves in air to magnetic waves in the Sun's atmosphere. On the other hand, waves on a stretched elastic thread are the easiest systems to picture and explain.

The string is initially horizontal and has two fixed ends, let's say a left end (L) and a right end (R): When we shake the string from end L onward, we see a wave propagate across the string. The goal is to attempt to calculate the string's vertical displacement from the X-axis, u(x,t), as a function of location X. Additionally, timet: In other words, the displacement from equilibrium at Position X and time t is represented as u(x,t): A small portion of the string moved between Points P and Q is

Where



- The angle formed by the string and a horizontal line at point x and time t is denoted by θ(x,t).
- The string's tension at position x and time t is given by T(x,t);
- The mass density of the string at location x is given by ρ(x):

To get the wave equation, a few simplifying assumptions have to be made: The mass of the string between points P and Q is equal to ρ times its length, where Δ is the string's length and is defined by (7) since the string's density, ρ, stays constant.

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta u)^2} = \Delta x \sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2} \approx \Delta x \sqrt{\left(\frac{\partial u}{\partial x}\right)^2} \ldots .(8)$$

It is expected that the displacement, u(x,t), and its derivatives are tiny in order that

$$\Delta s \approx \Delta x \ldots ..(9)$$

and the mass of the portion of the string is $\rho\Delta x$

We may now separate the forces into their vertical and horizontal components.- Horizontal: The small string's net horizontal force is

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t)-T(x,t)\cos\theta(x,t) \text{ .....(10)}$$

As there isn't any horizontal motion, we have to

$$T(x,t)\cos\theta(x,t)=T(x+\Delta x,t)\cos\theta(x+\Delta x,t)=T \text{ ......(11)}$$

Vertical: At P the tension force is $-T(x,t)\sin\theta(x,t)$ where as at Q the Force is $T(x+\Delta x,t)\sin\theta(x+\Delta x,t)$.

Newton's Law of Motion follows.

Acceleration of mass applied forces provides

$$\rho\Delta x\frac{\partial^2 u}{\partial t^2}=T(x+\Delta x,t)\sin\theta(x+\Delta x,t)-T(x,t)\sin\theta(x,t).$$

$$\frac{\rho}{\rho}\Delta x\frac{\partial^2 u}{\partial t^2}=\frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t)}{T(x+\Delta x,t)\sin\theta(x+\Delta x,t)}=-\frac{T(x,t)\sin\theta(x,t)}{T(x,t)\cos\theta(x,t)}$$

$$\tan\theta(x+\Delta x,t)-\tan\theta(x,t) \text{ ......(12)}$$

But

$$\tan\theta(x,t)=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=u_x(x,t).$$

$$\tan\theta(x+\Delta x,t)=u_x(x+\Delta x,t)\text{....(13)}$$

we get

$$\frac{\rho}{T}\Delta x u_{tt}(x,t)=u_x(x+\Delta x,t)-u_x(x,t)\text{.....(14)}$$

Separating by Δx and letting $\Delta x\to 0$

$$\frac{\rho}{T}\Delta x u_{tt}(x,t)=u_{xx}(x,t)$$

$$u_{tt}(x,t)=c^2u_{xx}(x,t)\text{......(15)}$$

Where $c^2=\frac{T}{\rho}$,

This partial differential equation represents the string's transverse vibration. The one-dimensional wave equation is another name for it.

6. Homogeneous one-dimensional wave equation

Real-world physical conditions typically occur at fixed intervals. For two reasons, we can justify taking x on the entire real line. From a physical standpoint, the border won't affect you much if you are seated far away from it; however, the solutions found in this chapter are still applicable up until that point. The lack of a boundary is a significant simplification mathematically. Without the complexities of boundary conditions, the most basic properties of the PDEs can be discovered with the greatest ease.

$$u_{tt} = c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0 \ldots\ldots.(16)$$
$$u(x,0) = f(x) \text{ for } -\infty < x < \infty$$
$$u_t(x,0) = g(x) \text{ for } -\infty < x < \infty$$

Solution via change of variable

The equation is hyperbolic, thus we establish a new variable ε, η by

$$\begin{matrix} \varepsilon = x + ct \\ \eta = x - ct \end{matrix} \ldots.(17)$$

we have

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial \varepsilon} \cdot \frac{\partial \varepsilon}{\partial \eta} + \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} \\ &= C\frac{\partial}{\partial \varepsilon} + \frac{\partial}{\partial \eta}(-C) \end{aligned} \ldots\ldots(18)$$

$$C\left(\frac{\partial}{\partial \varepsilon} - \frac{\partial}{\partial \eta}\right)$$

$$\begin{aligned} \frac{\partial^2}{\partial t^2} &= C\left(\frac{\partial}{\partial \varepsilon} - \frac{\partial}{\partial \eta}\right) C\left(\frac{\partial}{\partial \varepsilon} - \frac{\partial}{\partial \eta}\right) \\ \frac{\partial^2}{\partial t^2} &= C^2\left(\frac{\partial}{\partial \varepsilon} - \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \varepsilon} - \frac{\partial}{\partial \eta}\right) \\ \frac{\partial^2}{\partial t^2} &= C^2\left(\frac{\partial^2}{\partial \varepsilon^2} - \frac{2\,\partial^2}{\partial \varepsilon\,\partial \eta} + \frac{\partial^2}{\partial \eta^2}\right) \end{aligned} \ldots..(19)$$

Likewise

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \varepsilon} \cdot \frac{\partial \varepsilon}{\partial x} + \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \ldots\ldots(20)$$

From the coordinate equations above, we have

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \varepsilon} \cdot 1 + \frac{\partial}{\partial \eta} \cdot 1 \\ \frac{\partial}{\partial x} &= \frac{\partial}{\partial \varepsilon} + \frac{\partial}{\partial \eta} \end{aligned} \ldots..(21)$$

So,

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{\partial \eta}\right) \\ &= \frac{\partial^2}{\partial \varepsilon^2} + \frac{2\,\partial^2}{\partial \varepsilon\,\partial \eta} + \frac{\partial^2}{\partial \eta^2} \end{aligned} \ldots..(22)$$

s.t.

$$U(x,t) = W(\varepsilon, \eta)$$

Later

$$U_{tt} - C^2 U_{xx} = 0 \ldots\ldots(23)$$

Converts

$$C^2\left(W_{\varepsilon\varepsilon} - 2W_{\varepsilon\eta} + W_{\eta\eta}\right) - C^2\left(W \quad + 2W_{\varepsilon\eta} + W_{\eta\eta}\right) = 0$$
$$\Rightarrow -4C^2 W_{\varepsilon\eta} = 0 \ldots\ldots.(24)$$
$$\Rightarrow W_{\varepsilon\eta} = 0$$

$$\frac{\partial^2 w}{\partial\varepsilon\,\partial\eta} = 0 \ldots\ldots..(25)$$

It is easy to find the general solution of the equation (4) by integrating it twice.

First suppose you integrate with respect to E and notice that the constant of integration must depend on η to get,s

$$\frac{\partial w}{\partial \eta} = G(\eta)\ldots\ldots.(26)$$

Integrate now with regard to η and note that ε determines the integration constant. $w = \int_0^\eta G(\eta)d\eta + F(\varepsilon)$…..(27)

let

$$\int_0^\eta G(\eta)d\eta = G(\eta)\ldots.(28)$$

So that

$$w = G(\eta) + F(\varepsilon)\ldots..(29)$$

Memory that we misshapen

$$U(x,t) = w(\varepsilon,\eta)\ldots..(30)$$

So

$$U(x,t) = F(x+ct) + G(x-ct)\ldots..(31)$$

Given;

$$u(x,0) = f(x) \text{ for } -\infty < x < \infty$$

$$u_t(x,0) = g(x) \text{ for } -\infty < x < \infty\ldots(32)$$

$$t = 0$$

$$U(x,0) = f(x) = F(x) + G(x)$$

$$U(x,0) = F(x) + G(x) = f(x)\ldots.(33)$$

Next, we distinguish (9) with regard to t using the chain rule, setting t=0 to obtain

$$U_t(x,0) = CF'(x) + CG'(x)\ldots..(34)$$

$$CF'(x) - CG'(x) = g(x)$$

$$C(F'(x) - G'(x)) = \frac{g(x)}{C}\ldots\ldots(35)$$

$$F(x) - G(x) = \int_0^x \frac{g(s)}{C}ds + k\ldots.(36)$$

$$2F(x) = \int_0^x \frac{g(s)}{C}ds + k + f(x)$$

$$F(x) = \frac{1}{2}\int_C^x g(s)ds + \frac{k}{2} + \frac{f(x)}{2}\ldots.(37)$$

Also subtracting

$$2G(x) = -\frac{1}{c}\int_0^x g(s)ds - k + f(x)$$

$$G(x) = -\frac{1}{2c}\int_0^x g(s)ds - \frac{k}{2} + \frac{f(x)}{2} \ldots..(38)$$

Recall that

$$U(x,t) = F(x+ct) + G(x-ct)$$

$$F(x+ct) = \frac{f(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} g(s)ds + \frac{k}{2} \ldots.(39)$$

And

$$G(x-ct) = \frac{f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^0 g(s)ds\frac{k}{2}$$

$$U(x,t) = f(x+ct) + f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \ldots..(40)$$

Thus d'Alembert's formula represents the unique solution of above equations.

7. Result

Considering equation (9) obtained that is,

$$U(x,t) = f(x+ct) + f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \ldots\ldots(41)$$

and the standard wave equation;

$$\mathrm{u_{tt}} = \mathrm{c^2 u_{xx}} \text{ for } -\infty < x < \infty, t > 0$$

$$u(x,0) = f(x) \text{ for } -\infty < x < \infty$$

$$u_t(x,0) = g(x) \text{ for } -\infty < x < \infty$$

Let solve some problems;

Problematic (1)

$$U_{tt} - 25U_{xx} = 0 \quad -\infty < x < \infty, \quad t > 0$$

$$U(x,0) = f(x) \quad -\infty < x < \infty,$$

$$U(x,0) = \sin x \quad -\infty < x < \infty, \ldots..(42)$$

$$U_t(x,0) = 0 \quad -\infty < x < \infty$$

Explanation

Comparing through the standard wave equation

$$U_{tt} - C^2U_{xx} = 0 \qquad -\infty < x < \infty, \quad t > 0$$

$$U(x,0) = f(x) \quad -\infty < x < \infty,$$

$$U_t(x,0) = g(x) \quad -\infty < x < \infty \ldots.(43)$$

Consequently from the above question

$$C^2 = 25 \quad \Rightarrow C = 5$$

$$f(x) = \sin x$$

$$g(x) = 0$$

Consequently, using

$$U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x)ds \ldots\ldots(44)$$

We need

$$U(x,t) = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} 0ds$$

$$U(x,t) = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + 0$$

$$U(x,t) = \frac{1}{2}[\sin(x+5t) + \sin(x-5t)] \ldots(45)$$

Recall that

$$\sin(A+B) + \sin(A-B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \ldots.(46)$$

Therefore, above equation yield

$$U(x,t) = \frac{1}{2}\left[2\sin\left(\frac{x-5t+x-5t}{2}\right)\cos\left(\frac{x+5t-x+5t}{2}\right)\right] \ldots\ldots(47)$$

Hereafter

$$U(x,t) = \sin x \cos 5t \ldots\ldots(48)$$

Table 1: viewing the standards of $u(x,t)$ at fluctuating x and t

S/N	$u(x,t)$	x	t
1	0.0672	5^0	0
2	0.1510	10^0	2
3	0.2332	15^0	4
4	0.2952	20^0	6
5	0.3227	25^0	8
6	0.3204	30^0	10
7	0.2858	35^0	12
8	0.2188	40^0	14
9	0.1218	45^0	16
10	0.010	50^0	18

Table 2: Viewing consequence of $u(x,t)$ at fixed x and changing t

S/N	$u(x,t)$	Fixed x	Varying t
1	0.0572	5	0
2	0.0849		2
3	0.0719		4
4	0.0745		6
5	0.0568		8

Problematic (2)

$$U_{tt} - 9U_{xx} = 0. \quad -\infty < x < \infty, \quad t > 0$$

$$U(x,0) = \sin x$$

$$U_t(x,o) = \cos 3x \ldots (49)$$

$$-\infty < x < \infty,$$

Explanation: Comparing through the standard wave equation

$$U_{tt} - C^2 U_{xx} = 0$$

$$-\infty < x < \infty, \quad t > 0$$

$$U(x,0) = f(x)$$

$$-\infty < x < \infty,$$

$$U_t(x,o) = g(x)$$

$$-\infty < x < \infty$$

Somewhere

$$C^2 = 9 \quad \Rightarrow C = 3$$

$$f(x) = \sin x$$

$$g(x) = \cos 3x$$

Consequently, using the formula

$$U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x)ds \ldots .(50)$$

We need

$$U(x,t) = \frac{1}{2}[\sin(x+3t) + \sin(x-3t)] + \frac{1}{2 \times 3}\int_{x-ct}^{x+ct} \cos 3s ds$$

Nevertheless

$U(x,t) = \frac{1}{2}[\sin(x + c3t) + \sin(x - 3t)] = \frac{2\sin x \cos 3t}{2}$….(51)

So

$$\frac{2\sin x \cos 3t}{2} + \frac{1}{6}\int_{x-ct}^{x+ct} \cos 3st$$

$$= \sin x \cos 3t + \frac{1}{18}[\sin 3t \cos x]\text{…..(52)}$$

Later

$$U(x,t) = \sin x \cos 3t + \frac{1}{9}\sin 3t \cos x\text{…..(53)}$$

Table 3: viewing the standards of $u(x,t)$ at changing x and t

S/N	$u(x,t)$	x	t
1	0.6772	5^0	0
2	0.1830	10^0	2
3	0.2744	15^0	4
4	0.3476	20^0	6
5	0.4260	25^0	8
6	0.4711	30^0	10
7	0.6165	35^0	12
8	0.5336	40^0	14
9	0.5305	45^0	16
10	0.5081	50^0	18

Table 4: Viewing consequence of $u(x,t)$ at fixed t and changing x

S/N	$u(x,t)$	Varying x	Fixed t
1	0.0862	5	0
2	0.1636	10	
3	0.2578	15	

4	0.3430	20	
5	0.4226	25	
6	0.5120	30	
7	0.5726	35	

8. Conclusion

There are numerous physical applications for the wave equation, ranging from sound waves in air to magnetic waves in the Sun's atmosphere. On the other hand, waves on a stretched elastic thread are the easiest systems to picture and explain. For two reasons, we can justify taking x on the entire real line. From a physical standpoint, the border won't affect you much if you are seated far away from it; however, the solutions found in this chapter are still applicable up until that point. The lack of a boundary is a significant simplification mathematically. The one-dimensional wave equation, or governing partial differential equation, depicts the transverse vibration of an elastic string (King and Billingham, 2000). The change of variable method has been used to obtain the analytical answer. One of the most important mathematical puzzles of the middle of the eighteenth century was the solution to the wave equation. Though the classical theory of partial differential equations deals almost completely with the well-posed, ill posed problems can be mathematically and scientifically interesting.

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