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A STUDY ON THE FORMULATION OF RIEMANNIAN MANIFOLD

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ABSTRACT

Within the confines of this investigation, it is possible to establish a Lie-Trotter formula for generic Riemannian manifolds. A formula that is predicated on the local presence of a midpoint operation is described by these phrases with regard to its formula. As a consequence of this, it is possible to demonstrate that continuous mappings over Riemannian manifolds that maintain the local midpoint structure are smooth. This is particularly true with regard to the situations in which (local) isometries take place. Through our research, we have developed a method that does not use sliding or twisting in order to define the rolling of a pseudo-Riemannian manifold over another manifold. We compare the definition of rolling solely by the intrinsic data, which is defined by the metric tensors on manifolds, with the concept of rolling without sliding and twisting of two isometrically embedded manifolds in a pseudo-Euclidean space. We seek for parallels and differences between the two definitions, and we compare and contrast them. Building takes place on the smooth distribution of the configuration space, which is a representation of the kinematic constraints of not sliding and not twisting. On top of that, this presentation offers a multitude of findings about the causative nature of the rolling curves. A number of examples are provided throughout the work in order to explain various topics and to assist in comprehending the consequences of the theoretical findings.

KEYWORD *Formulation, Riemannian, Manifold*

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INTRODUCATION

In order to ensure that supervised learning models are capable of performing the tasks for which they were designed, it is necessary to train them using a substantial quantity of examples that have been tagged., the process of recognising them may be a challenging, time-consuming, or expensive endeavour. The amount of effort that an experienced linguist will spend in order to correctly annotate spoken phrases in audio data is substantially more than the amount of time that is required to actually analyse the audio. When applied to a learning model, labelled samples that are not similar to one another have varied amounts of information and contribute different things to the learning process. This is because the different samples contain different amounts of information. The challenge that arises from this is both exciting and practical: how can we choose the examples that are the most valuable to label in a way that not only increases the learning rate of the model but also decreases the cost of labelling.

When it comes to learning, selecting the appropriate instances to categorise in statistics is pretty comparable to the traditional Design of Experiments. Traditional DOE was created for agricultural purposes to determine the link between several input elements and one output response. This was done with the intention of doing physical testing with minimal experimental resources. One of the most important and extensively investigated topics within the Department of Energy (DOE) is the formulation of optimality criteria for experimental design as well as methodologies for the construction of such designs. One possible way to characterise this method is by using the acronym "Optimal Design of Experiments" (ODOE). The objective of the ODOE is to develop experimental designs that are faultless by taking into consideration a statistical criterion. As a general rule, it is considered that the model is linear, and statistical criteria in classic ODOE theory are often linked with the estimation or prediction of model parameters. When dealing with DOE concerns, the standard operating procedure (OPS) often entails making the assumption that the experimental region is flat and consists of a limited number of significant variables, which are generally referred to as "factors." In contrast, the dimensionality of data that is either textual or image-based is often far higher than the number of variables that are included in a typical industrial or agricultural experiment. Examples of

modern learning issues that are included in this area include image recognition and text categorization, to name just two examples.

In this kind of environment, a manifold hypothesis is developed in order to complete a statistical learning job. The relevant training data are placed in a high-dimensional ambient space; yet, according to this theory, they should really be situated on a manifold with a lower dimension. The typical linear dimensionality reduction strategy, on the other hand, is predicated on the assumption that the data will be contained inside a linear subspace that is based on key components. Working with data that has a high dimensionality may provide a number of situations in which the manifold hypothesis can be seen successfully. From the study conducted by Roweis and Saul (2000) and Tenenbaum et al. (2000) onward, a number of studies have shown that data with a high dimensionality, such as text or image data, often sits on a manifold that has a lower dimensionality. It often displays sparsity in the high-dimensional environment in which it is situated. The Department of Energy is sceptical about whether or not it will be possible to gather and label a sufficient number of training examples to cover the high-dimensional ambient environment of the whole plant. It is possible that it will be possible to choose points from a manifold space that has less dimensions if there is a denser collection of data points in this low-dimensional environment. Due to the fact that traditional DOE methodologies do not take into account these detailed characteristics of modern high-dimensional data, it is unfortunate.

Despite the fact that the data points are really situated on the surface of a Torus, which is a two-dimensional object, they are accessible in a Euclidean space that is three-dimensional. Figure 1(a) illustrates a quick and easy illustration of this. These two experimental designs are shown in Figures 1(b) and 1(c), respectively, and they were applied to this dataset. Because of these differences in design, the learning results will likewise be different from one another. Consequently, in order to get the highest possible level of learning performance, we need to be aware of how to locate the most suitable architecture that takes into account the complex structure of the data.

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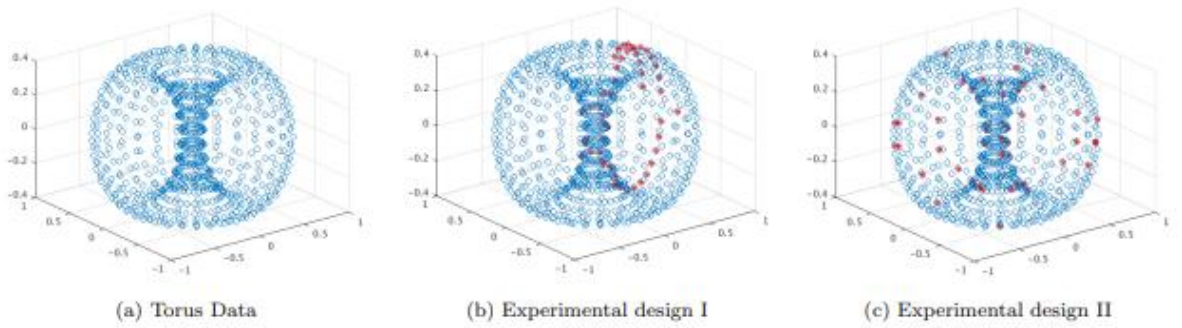

(a) Torus Data (b) Experimental design I (c) Experimental design II

Figure 1: While playing with a toy, data about a two-dimensional

It exhibits a torsus that is a component of a three-dimensional environment that surrounds it. In each of the studies, the red dots represent the samples that were selected for labelling purposes. The purpose of this research is to develop a theory and an algorithm that will allow for the construction of experiments that make use of high-dimensional manifold data in the most effective manner. In the course of this inquiry, the main objective is to collect as much valuable response data as possible while doing as few experimental runs as is practically practicable. It is going to be assumed that the training data has been collected on a Riemannian manifold with less dimensions. From a broader perspective, this manifold may be understood as a space that is curved and seems to be a Euclidean space when seen from a distance that is close enough to the neighbourhood. In the course of our investigation, we have not come across any previous literature that provides theoretical guarantees for experimental settings that are comparable to those on Riemannian manifolds. In spite of the fact that some authors have used ODOE criterion as a method of Active Learning in relation to data that has a great deal of dimensions, this is still the case.

The purpose of this paper is to provide a novel Equivalence Theorem for continuous optimal designs on manifold data. This theorem demonstrates that a D-optimal design and a G-optimal design on a Riemannian manifold are comparable to one another. We provide a new lower limit for the greatest prediction variance across the manifold by showing how a D/G optimum design may reach the lower bound. This allows us to provide a new lower bound. An method that we present, which we call ODOEM (optimum Design of Experiments on Manifolds), is designed to identify a continuous D/G optimum design on a Riemannian manifold and demonstrate that it

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will converge to the global D/G optimal design. This algorithm is a claim that we make about our whole body of work. In conclusion, we demonstrate that our ODOEM technique performs very well on a real-world picture dataset in addition to a generated manifold dataset (d).

Last but not least, this is the format that the remainder of the article follows. Following an introductory discussion of the traditional ODOE problem in Euclidean space, which is presented in Section 2, we proceed to demonstrate the developed manifold regularisation model.

By using this paradigm, which serves as the basis for our results, it is possible to get an understanding of the ODOE problem that pertains to manifolds. In addition, a unique equivalency theorem for Riemannian manifolds is described in the third part of the paper. In this section, our ODOEM approach is presented together with a theoretical justified explanation. While the ODOEM approach that was recommended is shown in the fourth section, the results of a convergence analysis are also presented. Following the conclusion of Section 5, many numerical tests demonstrate that the proposed method may be effectively applied to the task of finding optimal designs by making use of manifold data. We provide a concise summary as well as some suggestions for more study in the sixth part of this report.

Ideal Structure for Experiments on Manifolds

First, let's take a look at a model that uses linear regression.

$$y = f(x, \beta) + \varepsilon \; = \; \beta^{\top} g(x) + \varepsilon,$$

In the case when β is a column vector of unknown parameters, ε is assumed to have a distribution of N(0, σ2), and g: R d → R p is a nonlinear function that maps from the input space x ∈ R d to the feature space R p, the function is considered to be a vector mapping. The well-known ordinary least squares estimate of the β parameters may be obtained by providing a sample of n design points xi} n i=1 in the event that the matching response values yi} n i=1 are supplied:

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$$\hat{\beta} = \underset{\beta \in \mathbb{R}^p}{\operatorname{argmin}} \left\{ \sum_{i=1}^{n} (y_i - \beta^\top g(x_i))^2 \right\} = (X^\top X)^{-1} X^\top Y$$

where the i-th row of an n×p design matrix X is defined as $g(x_i)^\top$, in addition to Y, which is a response vector with dimensions of n×1. Because of this, the corresponding fitted function is considered to be $\hat{f}(x) = \hat{\beta}^\top g(x).$

A study of the traditional work on ODOE, which was pioneered.

For more references, check Pukelsheim (2006) and Fedorov and Leonov (2013) among others. An example of an optimality criteria for the linear regression model (1) is the D-optimality criterion, which minimises the determinant of the covariance matrix of the parameter estimates. This is only one example of an optimality criterion. $\operatorname{Var}(\hat{\beta}) = \sigma^2 (X^\top X)^{-1},$ as well as the maximum prediction variance, which is maxi=1,...,n, and is reduced by using the G-optimality criterion. $\{\operatorname{Var}(\hat{y}_i)\}.$ The term "alphabetic optimality" design criterion is used by Box and Draper (2007) to refer to factors such as these and other related criteria.

In spite of the fact that there have been some recent attempts to include alphabetic optimality constraints into manifold learning models (He, 2010; Chen et al., 2010; Aladdin et al., 2019), we are not aware of any theoretical framework that might potentially support these methods. It is only through the use of empirical testing that it is possible to guarantee that these techniques will be successful. It is thus necessary to construct a new theory of optimum experimental design, one that takes into consideration data on high-dimensional manifolds in a clear and unambiguous manner, makes an effort to explain existing techniques wherever it is feasible, and establishes a fundamental foundation for the creation of algorithms. Before we go on to the design of manifold experiments, it would be beneficial to provide a description of the manifold learning model that was developed by Belkin et al. (2006). A significant part will be played by this model in the subsequent events.

Model of Manifold Regularisation

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In accordance with the usual machine learning paradigm, learning is defined as the process of generating a function f: X → R that maps a data instance x to a label variable y by using the training data {xi} n i=1. An example of the collaboratively created labelled data set is denoted by the letter P $\{(x_i, y_i)\}_{i=1}^{l} \subset$ Both X × R and PX are representations of the marginal distribution, which is responsible for producing unlabeled data. $\{x_i\}_{i=l+1}^{n} \subset \mathcal{X} \subset$ In their study on expanding function learning across Riemannian manifolds with a wider range of dimensions, Belkin et al. (2006) make the premise that the conditional distribution $P(y|x)$ whenever x passes across a manifold that is supporting PX, it varies in a smooth manner. In a different way, if two data points are considered $x_1, x_2 \in \mathcal{X}$ are close to one another on this manifold, as judged by an intrinsic (or geodesic) distance, then the probability of the two labels are as follows: $P(y|x_1)$ and $P(y|x_2),$, which will be similar to one another. For the purpose of semi-supervised learning, these authors developed a method that requires the solution of the double regularized objective function that is shown below:

$$\hat{f} = \underset{f \in \mathcal{H}_K}{\operatorname{argmin}} \left\{ \sum_{i=1}^{l} V(x_i, y_i, f) + \lambda_A \|f\|_{\mathcal{H}_K}^2 + \lambda_I \|f\|_I^2 \right\}$$

a particular loss function, such as squared loss, is represented by the letter V. $(y_i - f(x_i))^2), \mathcal{H}_K$ is a Reproducing Kernel Hilbert Space (RKHS), which was developed by Aronszajn in 1950, paired with a Mercer kernel system. $\mathcal{K}, \|f\|_{\mathcal{H}_K}^2$ Wahba (1990) asserts that K is a penalty term that is associated with a Hong Kong standard and that functions to enforce smoothness criteria in the ambient space. $\|f\|_I^2$ PX's intrinsic manifold structure has a penalty term that is applied to it for the presence of non-smoothness along geodesics. As an additional point of interest, the degree of penalization in the ambient space and the intrinsic manifold that supports PX is regulated by two regularisation parameters, namely λA and λI, respectively. An alternative interpretation of the manifold regularisation model is that it is the spatial regression model that was provided by Ettinger et al. (2016), with the ambient space regularisation

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removed. The current work that has been done on non-Euclidean data methods is related to (3) in this manner, among other ways. In spite of the fact that there are numerous other nonparametric regression models on manifolds (for instance, Cheng and Wu, 2013; Marzio et al., 2014; Lin et al., 2017), the focus of this paper is on the manifold regularisation model proposed by Belkin et al. (2006). This model provides a helpful representer theorem, which will be further explained in the following paragraphs.

Naturally, the decision of $\|f\|_I^2$ Taking into consideration the probability distribution PX, I ought to be considered a smoothness penalty. On the other hand, since PX is unknown, the vast majority of applications that take place in the actual world need the use of empirical predictions of the marginal distribution. The situation in which PX is supported on a compact manifold M ⊂ R d has been the topic of a significant amount of research (Roweis and Saul, 2000; Tenenbaum et al., 2000; Belkin and Niyogi, 2003; Donoho and Grimes, 2003; Coifman et al., 2005). It is possible to demonstrate, with the help of this presumption, that problem number three may be reduced to (see Belkin, 2003; Lafon, 2004).

$$\hat{f} = \underset{f \in \mathcal{H}_K}{\operatorname{argmin}} \left\{ \sum_{i=1}^{l} V(x_i, y_i, f) + \lambda_A \|f\|_{\mathcal{H}_K}^2 + \lambda_I \mathbf{f}^\top L \mathbf{f} \right\}$$

where $\mathbf{f} = [f(x_1), ..., f(x_n)]^\top$ The Laplacian matrix, denoted by the letter L, is connected to the data adjacency graph, denoted by the letter G. This graph is constructed by using all of the data points, both labelled and unlabeled. $\{x_i\}_{i=1}^n$. To be more specific, the graph Laplacian L is a good approximation of the Laplace-Beltrami operator that operates on the continuous Riemannian manifold M (for more information, see Belkin and Niyogi, 2005; Coifman et al., 2005; Hein et al., 2005). When learning about manifolds, it is usual practice to utilise a graph and the geodesic distances that correlate to it as an approximate representation of the manifold M. In a theoretical sense, this is supported by the Laplacian's convergence of the graph, which provides a precise sense in which the graph approaches M as the number of data points increases in density. To put it another way, the formula f > Lf serves as a preliminary estimate

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for since it The penalty for the lack of smoothness of function f is applied whenever it swings across nearby points in graph G using this method. Expression of the answer to the infinite-dimensional issue may be accomplished by the use of a finite sum over the points that are named and those that are not labelled:

$$f(x) = \sum_{i=1}^{n} \alpha_i \mathcal{K}(x_i, x)$$

in which the Mercer kernel that is linked to the ambient space HK is represented by the notation K(•, •). This is comparable to a representer theorem in the theory of splines (Kimeldorf and Wahba, 1970; Wahba, 1990), and it is applicable to the fourth problem mentioned in the previous sentence.

Consistent ODOE on Manifolds

Vuchkov (1977) gave the first discussion of a regularised technique in the ODOE literature. This discussion was based on the ridge regression estimator:

$$\hat{\beta}_{\text{ridge}} = \underset{\beta \in \mathbb{R}^p}{\operatorname{argmin}} \left\{ \sum_{i=1}^{l} (y_i - \beta^\top g(x_i))^2 + \lambda_{\text{ridge}} \|\beta\|^2 \right\}$$

Vuchkov's objective was to address problems that are either one-of-a-kind or ill-conditioned and that occur as a result of the sequential application of a D-optimal design technique in situations where there are fewer design points than there are parameters to estimate by using the ridge estimator. In situations when V is a squared-loss function, the RKHS HK has a L 2 norm, and the manifold regularisation parameter λI is zero, the learning problem (4) is more comprehensive than the ridge solution.

Following the presentation of some terminology, we proceed to investigate the most effective experimental design for the wide manifold regularisation model (4). For the sake of generality, let us assume that you are confronted with a sequential experimental design problem. In this

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scenario, the series starts from the beginning, and there is no data that has been coloured. I beg you, $\{z_i\}_{i=1}^{k} \subset \{x_i\}_{i=1}^{n}$ represent the collection of labelled points at iteration k, and let y = (y1,..., yk) > represent the vector of labels or replies that corresponds to the collection. According to Belkin et al. (2006), the Laplacian Regularized Least Squares (Lapel's) problem is generated from the manifold regularisation model (4), which is used in conjunction with a square loss function:

$$\hat{f} = \underset{f \in \mathcal{H}_K}{\operatorname{argmin}} \left\{ \sum_{i=1}^{k} (y_i - f(z_i))^2 + \lambda_A \|f\|_{\mathcal{H}_\mathcal{K}}^2 + \lambda_I \mathbf{f}^\top L\mathbf{f} \right\}.$$

Algorithm Proposal and Convergence Analysis

There are a few more discoveries that need to be made before we can proceed with the analysis of the proposed method for selecting the most effective experimental design for manifolds. These findings and the proofs that support them may be found in the appendix.

Proposition 5. $Let\ M_{Lap}(\epsilon_k)$ be the design's information matrix $\epsilon_k\ at\ k\text{-}th$ h iteration. $Let\ M_{Lap}(\epsilon(z))$ the design's information matrix focused at a single location, z. Given $\epsilon_{k+1} = (1-\alpha)\epsilon_k + \alpha\epsilon(z),$, then

$$|M_{Lap}(\epsilon_{k+1})| = (1-\alpha)^p \left| M_{Lap}(\epsilon_k) \right| \left[1 + \frac{\alpha}{1-\alpha} d(z, \epsilon_k) + \frac{\alpha}{1-\alpha} \operatorname{Tr}(M_{Lap}^{-1}(\epsilon_k) C) \right]$$

Riemannian geometry is built on the foundation of three fundamental ideas. According to N. I. Lobachevski, the earliest of these discoveries was the realization that there is a geometry that is not Euclidean. The idea of the interior geometry of surfaces, which was proposed by C. F. Gauss, is the second, and the idea of an n-dimensional space, which was proposed by B. Riemann, is the third. The third case study includes this paper as one of its components. The idea of Riemannian geometry had a significant impact on Albert Einstein's general theory of relativity, which was greatly affected by it. The manifold is regarded to be quasi-Einstein if the

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Ricci tensor S of a non-flat, n-dimensional Riemannian manifold (Mn, g), where n is more than two, is not identically zero and satisfies the tensorial formula.

$$S(X,Y) = ag(X,Y) + b\pi(X)\pi(Y), \qquad X,Y \in TM$$

with respect to the smooth functions a and b(6=0), where is a form that is not zero and is connected with the unit vector field and the Riemannian metric g. One of the forms is the associated one of the manifolds that contains the generator, which is the unit vector field. Utilizing the time-like velocity vector field of the ideal fluid as its foundation, the manifold generator is constructed. We have discovered that a four-dimensional semi-Riemannian quasi-Einstein manifold is capable of describing a collection of pressure-free ideal fluids that are described by general relativity and do not interact with one another. There are two scalars, r 2 and, that are connected to this manifold. Both of these connections are related. The symbols r represents energy density and scalar curvature, respectively, in this context. Manifolds that are locally product manifolds of one-dimensional distributions are referred to as quasi-Einstein manifolds in the mathematical lexicon. U and a distribution with (n1) dimensions are the two possibilities. Before U to be considered involutive and integrable, it is necessary for the generator of the manifold to be a parallel vector field. There are two distinct eigenvalues for the Ricci tensor that are associated with an n-dimensional quasi-Einstein manifold. In this context, the eigenvalues that are being discussed are a and a plus b. The multiplicity of an is n minus one, but the multiplicity of a plus b is more fundamental. The emergence of a quasi-Einstein manifold occurs naturally as a result of the existence of a genuine ξ−Einstein contact metric manifold. Some of the geometrical and physical features that are shown by quasi-Einstein manifolds are rather extensive.

A Riemannian manifold Mn's k−nullity distribution N(k) is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k\,[g(Y,Z)X - g(X,Z)Y]\}$$

with respect to any vector fields X, Y, and Z, where k is a smooth function on Mn and R is the Riemannian curvature tensor. According to reference the quasi-Einstein manifold is referred to

as a N(k)−quasi-Einstein manifold when the generator ξ of the manifold Mn is a member of the k−nullity distribution.

It is essential to have a space that has a continuous curvature in order to build differential geometry, mathematical physics, and mechanics. Cartan (1926) is credited with establishing the idea of a locally symmetric Riemannian manifold, which is a logical extension of manifolds that have a constant curvature. The criterion for local symmetry may be described in the following manner: the local geodesic symmetry F(x) is an isometry at each and every point x that belongs to the range M. The notion of a locally φ−symmetric Sasakian manifold was first introduced by Takahashi in the year 1977. Since that time, a number of geometers have conducted research on the properties of manifolds of this kind on various spaces.
In the year 1924, Friedmann and Schouten were the ones who first put up the concept of a semi-symmetric linear connection acting on a differentiable manifold. Hayden was the first person to introduce and develop the idea of a semi-symmetric linear connection with torsion on a Riemannian manifold. This notion was initially published and developed in 1932. During the year 1970, after a significant pause, Yano picked up where he left off with the systematic investigation of semi-symmetric metric connection on a Riemannian manifold. A number of publications, including investigate the characteristics of the semi-symmetric metric connection structures. The year 2001 saw the development and analysis of a projective semi-symmetric connection on a Riemannian manifold by P. Zhao and H. Song. The properties of this relationship have been investigated by a large number of scholars, including Zhao, Pal, Pandey, and Singh.

With the material that was previously investigated serving as a source of inspiration, the authors start looking into the properties of Riemannian manifolds that have a link that is projective and semi-symmetric. The following is the way that the present document is organized: In the second part, which follows the introduction, a brief summary of the projective semi-symmetric connection is presented. The projective curvature tensor of the Levi-Civita connection, denoted by Δ, is shown to coincide with the curvature tensor with regard to the projective semi-symmetric connection, denoted by $\tilde{\nabla}$. Furthermore, our findings provide evidence that the manifold (Mn, g) that has $\tilde{\nabla}$ is a particular kind of quasi-Einstein manifold. Furthermore, we

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establish that the characteristic vector field ξ is a member of the λ−nullity distribution that is associated with the connection ∇˜ itself. In Section 3, we conduct an analysis of the projective curvature tensor that is endowed with the projective semi-symmetric connection ∭˜. Furthermore, we establish that the projective curvature tensors coincide with respect to Δ and Δ. Within the fourth part, a comprehensive list of the features of a semi-symmetric Riemannian manifold that allows for a projective semi-symmetric connection ∇˜ is presented. It has been proven that the manifold is considered semi-symmetric for Δ˜ if and only if it is one-dimensional and flat. A Riemannian manifold with a projective semi-symmetric connection that fulfils the condition R. ˜ P، = 0 is the subject of the investigation that is provided in Section 5. Additionally, some surprising findings are also offered. In the last section, we establish a particular case that verifies the existence of this relationship ∭˜ and supports specific conclusions.

Examining the Required Notes

R stands for the Riemannian curvature tensor, Q for the Ricci operator, S for the Riemannian tensor, and r for the scalar curvature of a Riemannian manifold (Mn, g). The Ricci operator is also represented by the letter of the same name. In light of this, the following is how R, Q, S, and r are defined:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$QX = \sum_{i=1}^{n} R(X, e_i)e_i,$$

$$S(X,Y) = g(QX,Y) = \sum_{i=1}^{n} g\big(R(X, e_i)e_i, Y\big),$$

$$r = \sum_{i=1}^{n} S(e_i, e_i),$$

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where X, Y, and Z are points in the space Γ(TM), ∬ represents the Levi-Civita connection with respect to g, and ei} is an orthonormal frame. Every vector field X on M provides us with

$$X = \sum_{i=1}^{n} g(X, e_i)e_i,$$

A vector field V's divergence is determined by:

$$\mathrm{div}V = \sum_{i=1}^{n} g(\nabla_{e_i} V, e_i),$$

Hence one may get for a unit closed 1-form η (i.e., dη = 0)

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X), \qquad g(\nabla_X \xi, \xi) = 0 \qquad \text{and} \qquad \nabla_\xi \xi = 0.$$

Therefore, the g-dual of ξ is denoted by η, and for any vector fields X on M, η(X) is equal to g(X, ξ). The sole condition under which is deemed to be a vector field of the Jacobi type is if and only if,

$$\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi - R(X, \xi)Y = 0,$$

By utilising and substituting Y = ξ in one may derive,

$$\nabla_{\nabla_X \xi} \xi + R(X, \xi)\xi = 0,$$

A Riemannian manifold's scalar curvature, the Ricci operator, and the Riemannian curvature tensor are each represented by the letters R, Q, S, and r, respectively. In the following, we will define R, Q S, and r as follows:

Isometric immersions and Riemannian submanifolds

Considering a submanifold that is submerged, let (M, g) be a Riemannian manifold. $\iota : N \to M.$ One may deduce from this that δ represents an injective immersion, whereas N

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represents a smooth manifold. Following this, a Riemannian metric gN is induced by the Riemannian metric g in the set N. Assume that p is contained inside N. It is possible to see the tangent space TpN as a subspace of the tangent space TpM by using the injective map $d\iota_p : T_pN \to T_{\iota}(p)M.$ We define $(\dot{g_N})_p$ to be only g's confinement to this subspace, namely,

$$(g_N)_p(u, v) = g_{\iota(p)}(d\iota_p(u), d\iota_p(v)),$$

where $u, v \in T_pN.$ There is no doubt that gN is a Riemannian metric. A Riemannian submanifold of (M, g) is denoted by the notation (N, gN), while the induced Riemannian metric in N is denoted by the notation N.

Despite the fact that δ is an immersion that is not necessarily injective, it is important to remember that the definition of gN is still accurate and makes sense. In this context, we designate gN as the pulled-back metric, write gN = ι ∗ g, and assert that ι: (N, gN) → (M, g) is an isometric immersion. It is important to note that this is a local injective immersion. Within the realm of exceptional examples, the problem of Riemannian submanifolds of Euclidean space stands out as especially crucial. From a historical perspective, the study of Riemannian manifolds arose after the theory of curves and surfaces in R3, which began in the third century. Rather of using local charts, classical theory makes use of parameterizations. These objects are referred to as parameterized surfaces and curves due to the fact that they generally already have a parameterization. The assumption that the parameterization is smooth is only made in the most general of hypothetical scenarios. The term "regular surface" or "curve" is used when one desires for the parameterization to possess the quality of an immersion. The conclusion that can be drawn from this is that the parameterization in this particular case is, of course, an embedding. According to the classical theory, this is adequate for it since its primary interest is with computations that take place on a local scale.

CONCLUSIONS

On Riemannian manifolds, this research aims to provide theoretical underpinnings for optimum experimental designs. In the same way as in the Euclidean case, when the repressors are on a

manifold, D-optimal and G-optimal designs are equivalent. A new lower bound for the maximum prediction variance was also set by us, and we proved that the D/G optimum design satisfies this lower bound. To achieve this, a lower limit is imposed. To find the optimal manifold experimental configuration, we offered a convergent approach. Next, we set out to determine our ODOEM method's optimal performance by comparing it to other popular manifold and euclidean optimum design of experiments approaches using both simulated datasets and real-world picture scenarios. From this, we were able to deduce the best course of action to take as a whole. Other avenues of investigation could be opened up by this finding. Preliminary studies may provide a methodical approach to selecting the regularisation parameters λA and λI from the given equation. Due to the initial lack of or insufficient number of tagged instances, sequential learning problems cannot be solved via cross-validation. There are hardly many circumstances like this. Model selection criteria that include theoretical guarantees might improve learning performance, even when λA and λI are explained using heuristics. Among other things, the researchers found the best likelihood function to use for calculating λA and λI in a Gaussian Process model. Additionally, it is acknowledged that there are other optimality measures for experimental design outside the D/G criterion.

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