# The number of Smallest parts of *overpartitions* of $n$

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## ABSTRACT

Sylvie Corteel and Jeremy Lovejoy [7] defined *overpartitions* and George E Andrews derived formula for the number of smallest parts of *partitions* of a positive integer $n$. In this paper we derived the formula for the number of smallest parts of *overpartitions* of a positive integer $n$ by using the concepts of $r-overpartitions$.

**Keywords:** *partition, overpartition*, $r-overpartition$, smallest parts of the *partition* and $r-overpartition$ of positive integer $n$.

**Subject classification:** 11P81 Elementary theory of *partitions*.

**Introduction:**

An *overpartition* of $n$ [7] is a non increasing sequence of natural numbers whose sum is $n$ in which first (equivalently, the final) occurrence of a number may be *overlined*. We denote the number of *overpartitions* of $n$ by $\overline{p}(n)$. Since the *overlined* parts form a *partition* into distinct parts and the *non – overlined* parts form an *ordinary partition*, the generating function for the number of *overpartitions* is

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots$$

For example, the 14 *overpartitions* of 4 are

$4$, $\bar{4}$, $3+1$, $\bar{3}+1$, $3+\bar{1}$, $\bar{3}+\bar{1}$, $2+2$, $\bar{2}+2$, $2+1+1$, $\bar{2}+1+1$, $2+\bar{1}+1$, $\bar{2}+\bar{1}+1$, $1+1+1+1$, $\bar{1}+1+1+1$

Let $\overline{\xi}(n)$ denote the set of all *overpartitions* of $n$ and $\overline{p}(n)$ the cardinality of $\overline{\xi}(n)$ for $n \in N$. If $1 \le r \le n$ write $\overline{p_r}(n)$ for the number of *overpartitions* of $n$ each consisting of exactly $r$ parts, i.e $r-overpartitions$ of $n$. If $r \le 0$ or $r \ge n$ we write $\overline{p_r}(n) = 0$

If $r \le 0$ or $r \ge n$ we write $\overline{p_r(n)} = 0$ .Let $\overline{p(k,n)}$ represent the number of *overpartitions* of $n$ using natural numbers atleast as large as $k$ only.

Let $\overline{spt(n)}$ denote the number of smallest parts including repetitions in all *overpartitions* of $n$ . For $i \ge 1$ let us adopt the following notation.

$$m_s(\overline{\lambda}) = \text{number of smallest parts of } \overline{\lambda} \ .$$

$$\overline{spt(n)} = \sum_{\lambda \in \xi(n)} m_s(\overline{\lambda})$$

**1.1** Existing generating functions are given below.

| Function | Generating function |
|---|---|
| $p_r(n)$ | $\dfrac{q^r}{(q)_r}$ |

| | |
|---|---|
| $p_r(n-k)$ | $\dfrac{q^{r+k}}{(q)_r}$ |
| number of divisors | $\displaystyle\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)}$ |
| sum of divisors | $\displaystyle\sum_{n=1}^{\infty}\frac{n.q^n}{(1-q^n)}$ (1.1.1) |

where $(q)_k=\prod_{n=1}^{k}(1-q^n)$ for $k>o$, $(q)_k=1$ for $k=o$ and $(q)_k=0$ for $k<o$.

Since $(a)_n=(a;q)_n=(1-a)(1-aq)(1-aq^2)...(1-aq^{n-1})$ [1]

**1.2:** Derivation for $r-overpartitions$:

$$\overline{p_1(n)}=2p_1(n)=\frac{2q}{(1-q)}=\frac{q(-1,q)_1}{(q)_1}$$

$$\overline{p_2(n)}=2^2p_2(n)-2p_1\left(\left[\frac{n}{2}\right]\right)=\frac{2^2q^2}{(1-q)(1-q^2)}-\frac{2q^2}{(1-q^2)}$$

$$=\frac{2q^2}{(1-q^2)}\left\{\frac{2}{(1-q)}-1\right\}=\frac{2q^2(1+q)}{(1-q)(1-q^2)}=\frac{q^2(-1,q)_2}{(q)_2}$$

By induction, we get

$$\overline{p_r(n)}=\frac{q^r2(1+q)(1+q^2)...(1+q^{r-1})}{(1-q)(1-q^2)(1-q^3)...(1-q^r)}=\frac{q^r(-1,q)_r}{(q)_r}$$

and $\overline{p_r(n-a)}=\dfrac{q^{r+a}(-1,q)_r}{(q)_r}$ (1.2.1)

And also we observe that the generating function for the number of *overpartitions* is $\sum_{n=0}^{\infty}\overline{p(n)}q^n=\prod_{n=1}^{\infty}\frac{1+q^{n-1}}{1-q^n}$

**1.3 Theorem:**

$$\overline{spt(n)}=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{p(k,n-tk)}+\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{p(k+1,n-tk)}+2d(n) \quad (1.3.1)$$

**Proof :** [2] Let $n=(\lambda_1,\lambda_2,\ldots,\lambda_r)=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}},k^{\alpha_l}\right)$ be any $r-partition$ of $n$ with $l$ distinct parts. For corresponding to it there are $2^l$ times $r-overpartitions$ of *n*. (1.3.2)

**Case 1:**[3] Let $r>\alpha_l=t$ that means $\lambda_{r-t}>k$

Subtract all $k's$, we get $n-tk=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}}\right)$

Hence $n-tk=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}}\right)$ is a $(r-t)-partition$ of $n-tk$ with $l$- 1 distinct parts and each part greater than $k+1$. For corresponding to it they are $2^{l-1}$ times $(r-t)-overpartitions$ of $n-tk$. From (1.3.2) we know that the total number of $r-overpartitions$ are $2^l$.

Now we get, 2 times the number $\overline{p_{r-t}(k+1,n-tk)}$ of $r-overpartitions$ from $r-partitions$ of *n* with exactly $t$ smallest elements as *k*.

**Case 2:** Let $r>\alpha_l>t$ that means $\lambda_{r-t}=k$

Omit $k's$ from last $t$ places, we get $n-tk=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}},k^{\alpha_l-t}\right)$

Hence $n-tk=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}},k^{\alpha_l-t}\right)$ is a $(r-t)-partition$ of $n-tk$ with $l$ distinct parts and the least part is $k$. For corresponding to it, there are $2^l$ times of $r-overpartitions$ of $n-tk$ with least part $k$.

Now we get the number $\overline{f_{r-t}(k,n-tk)}$ of $r-overpartitions$ from a $r-partitions$ of $n$ with more than $t$ smallest elements as *k*.

**Case 3:** Let $r=\alpha_l=t$ that means all parts in the *partition* are equal. For each $r-partition$ with equal parts have 2 times of $r-overpartitions$ of *n*.

From cases (1), (2) and (3) we get $r-overpartitions$ of *n* with *t* smallest parts as *k* is

$$\overline{f_{r-t}(k,n-tk)}+2\overline{p_{r-t}(k+1,n-tk)}+2\beta$$

where $\beta=1$ if $r\mid n$ and $\beta=0$ otherwise

$$=\overline{f_{r-t}(k,n-tk)}+\overline{p_{r-t}(k+1,n-tk)}+\overline{p_{r-t}(k+1,n-tk)}+2\beta$$

$$=\overline{p_{r-t}\left(k,n-tk\right)}+\overline{p_{r-t}\left(k+1,n-tk\right)}+2\beta$$

The number of *partitions* of *n* with equal parts is equal to the number of divisors of *n*. Since the number of divisors of *n* is $d\left(n\right)$. Then the number of *overpartitions* of *n* with all parts are equal is $2d\left(n\right)$.

From [5] and [6], the number of smallest parts in *overpartitions* of *n* is

$$\overline{spt\left(n\right)}=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{p\left(k,n-tk\right)}+\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{p\left(k+1,n-tk\right)}+2d\left(n\right)$$

**1.4 Theorem:** $\overline{p_r\left(k+1,n\right)}=\overline{p_r\left(n-kr\right)}$ (1.4.1)

**Proof :** Let $n=(\lambda_1,\lambda_2,\ldots,\lambda_r),\lambda_i>k\ \ \forall i$ be any $r-overpartition$ of $n$.

Subtracting each part by $k$, we get $n-kr=\left(\lambda_1-k,\lambda_2-k,\ldots,\lambda_r-k\right)$

Hence $n-kr=\left(\lambda_1-k,\lambda_2-k,\ldots,\lambda_r-k\right)$ is a $r-overpartition$ of $n-kr$.

Therefore the number of $r-overpartitions$ of $n$ with parts greater than or equal to $k+1$ is $\overline{p_r\left(n-kr\right)}$

**1.5 Theorem:** $\sum_{n=0}^{\infty}\overline{spt\left(n\right)}\,q^n=\frac{\left(-1,q\right)_\infty}{\left(q\right)_\infty}\sum_{n=1}^{\infty}\frac{2q^n}{\left(1-q^n\right)}\frac{\left(q\right)_{n-1}}{\left(-1,q\right)_{n+1}}$

**Proof:** From theorem (1.3.1), we have

$$\overline{spt\left(n\right)}=\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\overline{p\left(k,n-tk\right)}+\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\overline{p\left(k+1,n-tk\right)}+2d\left(n\right)$$

Replace $k+1$ by $k$, $n$ by $n-tk$ for first part and $n$ by $n-tk$ for second part in (1.3.1)

$$=\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}\overline{p_r\left(n-tk-r\left(k-1\right)\right)}+\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}\overline{p_r\left(n-tk-rk\right)}+2d\left(n\right)$$

Where $d\left(n\right)$ is the number of positive divisors of $n$.

From (1.1.1)

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+tk+r\left(k-1\right)}\left(-1,q\right)_r}{\left(q\right)_r}+\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+tk+rk}\left(-1,q\right)_r}{\left(q\right)_r}+\sum_{k=1}^{\infty}\frac{2q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{tk+rk}(-1,q)_r}{(q)_r}+\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+tk+rk}(-1,q)_r}{(q)_r}+\sum_{k=1}^{\infty}\frac{2q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{tk}\left[\sum_{r=1}^{\infty}\frac{(q^k)^r(-1,q)_r}{(q)_r}\right]+\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{tk}\left[\sum_{r=1}^{\infty}\frac{q^r(q^k)^r(-1,q)_r}{(q)_r}\right]+\sum_{k=1}^{\infty}\frac{2q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\left[\left(1+\sum_{r=1}^{\infty}\frac{(q^k)^r(-1,q)_r}{(q)_r}\right)-1\right]$$

$$+\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\left[\left(1+\sum_{r=1}^{\infty}\frac{(q^{k+1})^r(-1,q)_r}{(q)_r}\right)-1\right]+\sum_{r=1}^{\infty}\frac{2q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\left(1+\sum_{r=1}^{\infty}\frac{(q^k)^r(-1,q)_r}{(q)_r}\right)+\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\left(1+\sum_{r=1}^{\infty}\frac{(q^{k+1})^r(-1,q)_r}{(q)_r}\right)$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\prod_{r=0}^{\infty}\left(\frac{1+q^rq^k}{1-q^rq^k}\right)+\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\prod_{r=0}^{\infty}\left(\frac{1+q^rq^{k+1}}{1-q^rq^{k+1}}\right)\qquad\text{from [1]}$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\prod_{r=0}^{\infty}\left(\frac{1+q^{r+k}}{1-q^{r+k}}\right)+\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\prod_{r=0}^{\infty}\left(\frac{1+q^{r+k+1}}{1-q^{r+k+1}}\right)$$

$$=\frac{(-1,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\frac{(q)_{k-1}}{(-1,q)_k}+\frac{(-1,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\frac{(q)_k}{(-1,q)_{k+1}}$$

$$=\frac{(-1,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\frac{(q)_{k-1}}{(-1,q)_k}\left[1+\frac{(1-q^k)}{(1+q^k)}\right]$$

$$=\frac{(-1,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{2q^k}{(1-q^k)}\frac{(q)_{k-1}}{(-1,q)_{k+1}}$$

$$=\frac{(-1,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{2q^n}{(1-q^n)}\frac{(q)_{n-1}}{(-1,q)_{n+1}}$$

**1.6 Corollary:** The generating function for the number $\overline{A_c(n)}$ of smallest parts of the *overpartitions* of $n$ which are multiples of $c$ is

$$\sum_{n=0}^{\infty}\overline{A_c(n)q^n}=\frac{(-1,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{2q^{cn}}{(1-q^{cn})}\frac{(q)_{cn-1}}{(-1,q)_{cn+1}}$$

**1.7 Corollary:** The generating function for the sum of smallest parts of the second *overpartitions* of $n$ is

$$\sum_{n=0}^{\infty}\overline{sum\,spt(n)q^{n}} = \frac{(-1,q)_{\infty}}{(q)_{\infty}}\sum_{n=1}^{\infty}\frac{2nq^{n}}{(1-q^{n})}\frac{(q)_{n-1}}{(-1,q)_{n+1}}$$

**Proof:** The generating function for the sum of smallest parts of the second *overpartitions* of a positive integer $n$ is

$$\begin{aligned}\overline{spt(n)} &= \sum_{r=1}^{\infty}\sum_{t=1}^{\infty} k\ \overline{p(k,n-tk)} + \sum_{r=1}^{\infty}\sum_{t=1}^{\infty} k\ \overline{p(k+1,n-tk)} + 2d(n)\\ &= \sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty} k\ \overline{p_{r}(n-tk-r(k-1))} + \sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty} k\ \overline{p_{r}(n-tk-rk)} + 2d(n)\end{aligned}$$

where $d(n)$ is the number of positive divisors of $n$.

From (1.1.1)

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{r+tk+r(k-1)}(-1,q)_{r}}{(q)_{r}}+\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{r+tk+rk}(-1,q)_{r}}{(q)_{r}}+\sum_{k=1}^{\infty}\frac{2kq^{k}}{1-q^{k}}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{tk+rk}(-1,q)_{r}}{(q)_{r}}+\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{r+tk+rk}(-1,q)_{r}}{(q)_{r}}+\sum_{k=1}^{\infty}\frac{2kq^{k}}{1-q^{k}}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}kq^{tk}\left[\sum_{r=1}^{\infty}\frac{(q^{k})^{r}(-1,q)_{r}}{(q)_{r}}\right]+\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}kq^{tk}\left[\sum_{r=1}^{\infty}\frac{q^{r}(q^{k})^{r}(-1,q)_{r}}{(q)_{r}}\right]+\sum_{k=1}^{\infty}\frac{2kq^{k}}{1-q^{k}}$$

$$=\sum_{k=1}^{\infty}\frac{kq^{k}}{(1-q^{k})}\left[\left(1+\sum_{r=1}^{\infty}\frac{(q^{k})^{r}(-1,q)_{r}}{(q)_{r}}\right)-1\right]$$

$$+\sum_{k=1}^{\infty}\frac{kq^{k}}{(1-q^{k})}\left[\left(1+\sum_{r=1}^{\infty}\frac{(q^{k+1})^{r}(-1,q)_{r}}{(q)_{r}}\right)-1\right]+\sum_{r=1}^{\infty}\frac{2kq^{k}}{1-q^{k}}$$

$$=\sum_{k=1}^{\infty}\frac{kq^{k}}{(1-q^{k})}\prod_{r=0}^{\infty}\left(\frac{1+q^{r}q^{k}}{1-q^{r}q^{k}}\right)+\sum_{k=1}^{\infty}\frac{kq^{k}}{(1-q^{k})}\prod_{r=0}^{\infty}\left(\frac{1+q^{r}q^{k+1}}{1-q^{r}q^{k+1}}\right)$$

$$=\sum_{k=1}^{\infty}\frac{kq^{k}}{(1-q^{k})}\prod_{r=0}^{\infty}\left(\frac{1+q^{r+k}}{1-q^{r+k}}\right)+\sum_{k=1}^{\infty}\frac{kq^{k}}{(1-q^{k})}\prod_{r=0}^{\infty}\left(\frac{1+q^{r+k+1}}{1-q^{r+k+1}}\right)$$

$$=\frac{(-1,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\frac{(q)_{k-1}}{(-1,q)_k}+\frac{(-1,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\frac{(q)_k}{(-1,q)_{k+1}}$$

$$=\frac{(-1,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\frac{(q)_{k-1}}{(-1,q)_k}\left[1+\frac{(1-q^k)}{(1+q^k)}\right]$$

$$=\frac{(-1,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{2kq^k}{(1-q^k)}\frac{(q)_{k-1}}{(-1,q)_{k+1}}$$

$$=\frac{(-1,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{2nq^n}{(1-q^n)}\frac{(q)_{n-1}}{(-1,q)_{n+1}}$$

$$\sum_{n=0}^{\infty}\overline{sum\,spt(n)q^n}=\frac{(-1,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{2nq^n}{(1-q^n)}\frac{(q)_{n-1}}{(-1,q)_{n+1}}$$

$$=\sum_{g_1=1}^{\infty}\left[\frac{g_1.q^{g_1}(-1,q)_{g_1}}{(q)_{g_1}}-\frac{2q^{g_1}}{(1-q^{g_1})}\sum_{g_2=1}^{g_1-1}\frac{g_2.q^{g_2}(-1,q)_{g_2}}{(q)_{g_2}}\right]$$

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