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OPTIMAL INVESTMENT STRATEGY FOR A CERTAIN CLASS OF PROBABILISTIC INVESTMENT PROBLEMS

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ABSTRACT

This article obtained the proof of the optimal investment strategy and corresponding rewards for a class probabilistic stationary investment problems, using backward dynamic programming recursive approach. In the sequel, the article formulated nontrivial extensions of the results to a larger dynamic class for practical and realistic considerations. The recursions were based on conditional probabilities and the proofs were achieved by deft deployment of probability axioms, set-theoretic facts, optimization and inductive principles. The extensions reflected and demonstrated consistency with the base results.

KEYWORDS: Dynamic Class, Dynamic Programming, Market Conditions, Optimal Investment Strategy, Policy Prescriptions, Probabilistic Investment Problems, Recursions, Stages, States, Stationary Class, Uncertainty.

1. Introduction

The concept of dynamic programming is largely based on mathematical recursions and the following Richard Bellman's principle of optimality as variously stated by Winston [1]: "Given the current state, the optimal decision must not depend on previously reached states or previously chosen decisions", Taha [2]: "Future decisions for the remaining stages will constitute an optimal policy regardless of the policy adopted in previous stages", Verma [3]: "An optimal policy (set of decisions) has the property that whatever be the initial stage and initial decisions, the remaining decisions must constitute an optimal policy for the state resulting from the first decisions.", Gupta and Hira [4]: "An optimal policy (a sequence of decisions) has the property that whatever the initial stage and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." In other words, suboptimal decisions taken at previous stages of a process do not preclude optimal decisions for the remaining stages of the process.

Probabilistic dynamic programming is a branch of Dynamic programming characterized by the uncertainty of states and returns at each stage. It arises for the most part in the treatment of stochastic inventory models and in Markovian decision processes. According to Wagner [5], the proper structuring of a model must take into account the intermingled sequence of decisions and emergent information about the exact values of the random elements. As asserted by [1], "in a probabilistic dynamic programming the decision maker's goal is to minimize expected (or expected discounted) cost incurred or to maximize expected (or expected discounted) reward earned over a given planning horizon.

The current investigation will use probabilistic dynamic programming to focus on the class of stationary investment problems, as formulated but not proved by [2]. Review of literature reveals the nonexistence of any formal proof of the prescribed investment strategy in [2]; needless to say that no extension of the formulation has been attempted by any author. This article fills these yawning gaps. The article will prove the results in [2] and obtain nontrivial extensions of those results to the dynamic class of investment problems, thus adding to the existing body of knowledge. For generalities on dynamic programming, see [1-5] and Taha [6]. For specialization of probabilistic dynamic programming to probabilistic differential dynamic programming- a powerful trajectory optimization approach with uncertainty in states and returns- see a recent paper by Pan and Theodorou [7].

2. Materials and Methods

2.1 The Stationary Investment Problem with Uncertainty

An individual wishes to invest up to C dollars in the stock market over the next n (years or periods). The investment plan calls for buying the stock at the start of the year (period) and selling it off at the end of the same year (period). Accumulated money may then be reinvested (in whole or part) at the start of the following year (period). The degree of risk in the investment is represented by expressing the return probabilistically. A study of the market shows that the return on investment is affected by m (favourable or unfavourable) market conditions and that condition k yields a return r_k with probability $p_k, k \in \{1, 2, \ldots, m\}$. How should the amount C be invested to realize the highest capital accumulation at the end of n time periods? Prove that the prescribed investment strategy is optimal.

2.2 Definition of investment capacities and decision variables

$x_i =$ Amount of funds available for investment at the start of period i. Note that $x_i = C$.
$y_i =$ Amount actually invested at the start of period i. Clearly, $y_i \le x_i$.

2.3 Elements of the Dynamic Programming Model

1. Stage i is represented by period i.
2. The alternatives at stage i are given by y_i.
3. The state at stage i is given by x_i.

2.4 Definition of the Backward Dynamic Programming Recursions

Let $f_i(x_i) =$ maximum expected funds for periods (years) $i, i+1, \ldots,$ and n, given x_i at the start of period i.

For market condition k we have the following relationship between stages i and $i+1$

a) $$x_{i+1} = \underbrace{x_i - y_i}_{\substack{\text{unutilized funds in period } i \\ \text{(carried over to period } i+1)}} + \underbrace{(1+r_k)y_i}_{\text{return on the 1-period investment}}$$

$$= x_i + r_k y_i$$

Given that market condition k occurs with probability $P_k; k \in \{1, 2, \ldots, m\}$ (m market contigencies), the maximal expected funds, $f_i(x_i)$ for periods $i, i+1, \ldots, n$ given that x_i monetary amount is available for investment at the start of periods i is defined recursively by reasoning as follows:

Expected fund (accumulated fund) from the beginning of period i to the end of period i is the expected available fund, x_{i+1} at the beginning of period $i+1$; $x_{i+1} = x_i + r_k y_i$, with probability p_k if market conditon k occurs. Therefore $p_k f_{i+1}(x_{i+1})$ is the expected fund for periods $i, i+1, \ldots, n$, given market condition k. Hence

$$f_i(x_i) = \max_{0 \le y_i \le x_i} \left\{ \sum_{k=1}^{m} p_k f_{i+1}\left(x_{i+1}\right) \right\} = \max_{0 \le y_i \le x_i} \left\{ \sum_{k=1}^{m} p_k f_{i+1}(x_i + r_k y_i); i = 1, 2, \ldots, n \right\}$$

$f_{n+1}(x_{n+1}) = x_{n+1}$, since no investment occurs at the end of period n and therefore no further funds (0 funds) are expected. x_{n+1} is the total availability at the beginning of period $n+1$ or at the end of period n, which is preserved.

Set $\bar{r} = \sum_{k=1}^{m} P_k r_k$: the expected market return from the m market contingencies.

Then the following theorem prescribes an optimal investment strategy.

3. Results and Discussion

3.1 Theorem 1: Optimal policy prescription for Stationary Probabilistic Investment Problems

For the general investment problem with m market conditions and horizon length n,

$$f_i(x_i) = \begin{cases} x_i, \text{ if } \bar{r} \le 0 \\ (1+\bar{r})^{n+1-i} x_i, \text{ if } \bar{r} > 0 \end{cases}$$

$$y_i^*(x_i) = y_i^* = \begin{cases} 0, \text{ if } \bar{r} \le 0 \\ x_i, \text{ if } \bar{r} > 0 \end{cases}$$

where y_i^* is the optimal amount invested at the start of period i.

In other words, for $i \in \{1, 2, \cdots, n\}$,

$$f_i(x_i) = x_i \operatorname{sgn}(\max\{1-\bar{r}, 0\}) + (1+\bar{r})^{n+1-i} x_i \operatorname{sgn}(\max\{\bar{r}, 0\}); \; y_i^* = x_i \operatorname{sgn}(\max\{\bar{r}, 0\}).$$

Proof

The proof is by mathematical induction on i. Computations are initiated from period n (stage n),

Thus,

$$f_n(x_n) = \max_{0 \le y_n \le x_n} \left\{ \sum_{k=1}^{m} p_k f_{n+1}(x_{n+1}) \right\} = \max_{0 \le y_n \le x_n} \left\{ \sum_{k=1}^{m} p_k x_{n+1} \right\} = \max_{0 \le y_n \le x_n} \left\{ \sum_{k=1}^{m} p_k (x_n + r_k y_n) \right\}$$

$$= \max_{0 \le y_n \le x_n} \left\{ x_n \sum_{k=1}^{m} p_k + \left(\sum_{k=1}^{m} p_k r_k \right) y_n \right\} = \max_{0 \le y_n \le x_n} \{ x_n + \bar{r} \, y_n \} = x_n + \max_{0 \le y_n \le x_n} \{ \bar{r} \, y_n \}$$

The maximum value occurs at $y_n = 0$, if $\bar{r} \le 0$ and at $y_n = x_n$, if $\bar{r} > 0$.

Therefore

$$y_n^* = \begin{cases} 0, \text{ if } \bar{r} \le 0 \\ x_n, \text{ if } \bar{r} > 0 \end{cases}$$

$$\Rightarrow f_n(x_n) = \begin{cases} x_n, \text{if } \bar{r} \le 0 \\ (1+\bar{r})\, x_n, \text{if } \bar{r} > 0. \end{cases}$$

So the theorem is true for $i = n.$

Assume that the theorem is valid for $j \le i \le n-1,$ for some positive integer $j > 1.$ Then in particular,

$$y_j^* = \begin{cases} 0, \text{ if } \bar{r} \le 0 \\ x_j, \text{if } \bar{r} > 0 \end{cases}$$

$$f_j(x_j) = \begin{cases} x_j \text{ if } \bar{r} \le 0 \\ (1+\bar{r})^{n+1-j}\, x_j, \text{if } \bar{r} > 0 \end{cases}$$

$f_{j-1}(x_{j-1})$ can now be effectively examined.

$$f_{j-1}(x_{j-1}) = \max_{0 \le y_{j-1} \le x_{j-1}} \left(\sum_{k=1}^{m} p_k\, f_j(x_j) \right) = \max_{0 \le y_{j-1} \le x_{j-1}} \left\{ \sum_{k=1}^{m} p_k \begin{cases} x_j, \text{if } \bar{r} \le 0 \\ (1+\bar{r})^{n+1-j} x_j \text{ if } \bar{r} > 0 \end{cases} \right\}$$

$$= \max_{0 \le y_{j-1} \le x_{j-1}} \begin{cases} \sum_{k=1}^{m} p_k \left(x_{j-1} + r_k y_{j-1} \right), \text{if } \bar{r} \le 0 \\ \sum_{k=1}^{m} p_k \left(1+\bar{r}\right)^{n+1-j} \left(x_{j-1} + r_k y_{j-1} \right), \text{if } \bar{r} > 0 \end{cases}$$

$$= \max_{0 \le y_{j-1} \le x_{j-1}} \begin{cases} x_{j-1} + \bar{r} y_{j-1} \text{ if } \bar{r} \le 0 \\ (1+\bar{r})^{n+1-j} \left(x_{j-1} + \bar{r} y_{j-1} \right), \text{if } \bar{r} > 0 \end{cases}$$

$$\Rightarrow y_{j-1}^* = \begin{cases} 0 \text{ if } \bar{r} \le 0 \\ x_{j-1} \text{ if } \bar{r} > 0 \end{cases}$$

$$\Rightarrow f_{j-1}\left(x_{j-1}\right) = \begin{cases} x_{j-1}, \text{ if } \bar{r} \le 0 \\ (1+\bar{r})^{n+1-j} (1+\bar{r})\, x_{j-1} \end{cases} \Rightarrow f_{j-1}\left(x_{j-1}\right) = \begin{cases} x_{j-1} \\ (1+\bar{r})^{n+1-(j-1)}\, x_{j-1}, \text{ if } \bar{r} > 0 \end{cases}$$

Therefore, the theorem is valid for $i = j-1$ and hence valid for all $1 \le i \le n.$ This completes the proof.

The ensuing theorem extends theorem 1 to the complex case where the probabilities and market returns vary from period to period; that is, for arbitrary number of different market conditions for each period with different probabilities and corresponding returns.

3.2 Theorem 2: The Optimal Policy Prescription for Dynamic Probabilistic Investment Problems

For the general investment problem with an arbitrary number of different market conditions for each period and corresponding returns, define the following:

$m_i =$ Number of market conditions in year i

$n =$ Horizon length

$r_{k_i} =$ Market return for market condition k_i in period i (stage i)

$p_{k_i} =$ Probability of market condition k_i in period i

$$\bar{r}_i = \sum_{k_i=1}^{m_i} p_{k_i} r_{k_i}, i \in \{1,\cdots,n\}; \bar{R}_i = \left\{\bar{r}_i,\ \bar{r}_{i+1},\cdots,\bar{r}_n\right\}, i \in \{1,2,\cdots,n-1\}; \bar{R}_n = \bar{r}_n;$$

$$\bar{R}_i^+ = \left\{\bar{r} \in \bar{R}_i : \bar{r} > 0\right\}; \bar{R}_i^- = \left\{\bar{r} \in \bar{R}_i : \bar{r} \le 0\right\}. \ \left(\text{Clearly } \bar{R}_i = \bar{R}_i^- \cup \bar{R}^+ \text{ and } \bar{R}_i^- \cap \bar{R}^+ = \varnothing\right)$$

Then subject to the standing hypotheses, the optimal investment strategy and the corresponding optimal return are prescribed as follows:

(a) $\quad y_i^* = x_i \operatorname{sgn}\left(\max\left\{\bar{r}_i, 0\right\}\right), i \in \{1,2,\cdots,n\}$

(b) $\quad f_i(x_i) = \prod_{\bar{r} \in \bar{R}_i^+} \left(1+\bar{r}\right) x_i$, if $\bar{R}_i^+$ is nonempty

(c) $\quad f_i(x_i) = x_i$, if if $\bar{R}_i^+$ is a null set $\left(\text{Equivalently } \bar{R}_i = \bar{R}_i^-\right)$,

where y_i^* is the optimal investment strategy at the start of period $i; i \in \left\{1,2,\ldots,n\right\}$. $f_i(x_i)$ is the maximal expected funds for periods $i, i+1,\ldots,n,$ given that the amount x_i is available for investment at the start of period i.

Proof of theorem 2

The dynamic programming recursive proof is inductive and is initiated from stage n.

3.3 Stage *n* Computations

$$x_{i+1} = x_i + r_{k_i} y_i; i \in \left\{1,2,\ldots,n\right\}$$

$$\Rightarrow f_n(x_n) = \max_{0 \le y_n \le x_n} \left\{\sum_{k_n=1}^{m_n} p_{k_n} \left(x_n + r_{k_n} y_n\right)\right\}$$

$$= \max_{0 \le y_n \le x_n} \left\{x_n + \sum_{k_n=1}^{m_n} \left(p_{k_n} r_{k_n}\right) y_n\right\} = \max_{0 \le y_n \le x_n} \left\{x_n + \bar{r}_n y_n\right\} = x_n + \max_{0 \le y_n \le x_n} \left\{\bar{r}_n y_n\right\}$$

$$\Rightarrow \quad y_n^* \quad = \quad \begin{cases} 0 \ \text{ if } \ \bar{r}_n \le 0 \\ x_n, \ \text{ if } \ \bar{r}_n > 0 \end{cases}$$

and

$$f_n\left(x_n\right) \quad = \quad \begin{cases} x_n \quad \text{if } \ \bar{r}_n \le 0 \\ \left(1+\bar{r}_n\right) x_n, \text{if } \ \bar{r}_n > 0 \end{cases} \quad = \quad \begin{cases} x_n \ \text{ if } \ \bar{r}_n \le 0 \\ \left(1+\bar{r}_n\right) x_n, \ \text{ if } \ \bar{r}_n > 0 \end{cases}$$

So the theorem is valid for $i = n$.

3.4 Stage n–1 Computations

$$x_{i+1} = x_i + r_{k_i} y_i; \, i \in \{1, 2, \ldots, n\}$$

$$\Rightarrow f_{n-1}(x_{n-1}) = \max_{0 \le y_{n-1} \le x_{n-1}} \left\{ \sum_{k_{n-1}=1}^{m_{n-1}} p_{k_{n-1}} f_n\left(x_n\right) \right\} = \max_{0 \le y_{n-1} \le x_{n-1}} \left\{ \sum_{k_{n-1}=1}^{m_{n-1}} p_{k_{n-1}} \begin{cases} x_n, \ \text{if } \ \bar{r}_n \le 0 \\ \left(1+\bar{r}_n\right) x_n, \text{if } \ \bar{r}_n \ge 0 \end{cases} \right\}$$

$$= \max_{0 \le y_{n-1} \le x_{n-1}} \left\{ \begin{cases} \sum_{k_{n-1}=1}^{m_{n-1}} p_{k_{n-1}} x_n, \ \text{if } \ \bar{r}_n \le 0 \\ \sum_{k_{n-1}=1}^{m_{n-1}} p_{k_{n-1}} \left(1+\bar{r}_n\right) x_n, \text{if } \ \bar{r}_n \ge 0 \end{cases} \right\} = \max_{0 \le y_{n-1} \le x_{n-1}} \left\{ \begin{cases} \sum_{k_{n-1}=1}^{m_{n-1}} p_{k_{n-1}} \left(x_{n-1} + r_{k_{n-1}} y_{n-1}\right), \ \text{if } \ \bar{r}_n \le 0 \\ \sum_{k_{n-1}=1}^{m_{n-1}} p_{k_{n-1}} \left(1+\bar{r}_n\right)\left(x_{n-1} + r_{k_{n-1}} y_{n-1}\right), \text{if } \ \bar{r}_n \ge 0 \end{cases} \right\}$$

$$= \max_{0 \le y_{n-1} \le x_{n-1}} \left\{ \begin{cases} x_{n-1} + y_{n-1}\bar{r}_{n-1}, \ \text{if } \ \bar{r}_n \le 0 \\ \left(1+\bar{r}_n\right) x_{n-1} + \left(1+\bar{r}_n\right) y_{n-1}\bar{r}_{n-1}, \ \text{if } \ \bar{r}_n > 0 \end{cases} \right\}$$

$$\Rightarrow \quad y_{n-1}^* \quad = \quad \begin{cases} 0 \ \text{ if } \ \bar{r}_{n-1} \le 0 \text{ and } \ \bar{r}_n \le 0 \\ x_{n-1}, \ \text{ if } \ \bar{r}_{n-1} > 0 \text{ and } \ \bar{r}_n > 0 \\ x_{n-1}, \ \text{ if } \ \bar{r}_{n-1} > 0 \text{ and } \ \bar{r}_n \le 0 \\ 0, \ \text{ if } \ \bar{r}_{n-1} \le 0 \text{ and } \ \bar{r}_n > 0 \end{cases}$$

$$\Rightarrow \quad y_{n-1}^* \quad = \quad \begin{cases} 0 \ \text{ if } \ \bar{r}_{n-1} \le 0 \\ x_{n-1}, \ \text{ if } \ \bar{r}_{n-1} > 0 \end{cases}$$

$$\Rightarrow \quad y_{n-1}^* \quad = \quad x_{n-1} \operatorname{sgn}\left(\max\left\{\bar{r}_{n-1}, 0\right\}\right)$$

$$\Rightarrow f_{n-1}(x_{n-1}) = \begin{cases} x_{n-1} \quad \text{if } \bar{r}_{n-1}, \bar{r}_n \le 0 \\ (1+\bar{r}_n)(1+\bar{r}_{n-1})x_{n-1}, \text{if } \bar{r}_{n-1}, \bar{r}_n > 0 \\ (1+\bar{r}_n)x_{n-1}, \text{if } \bar{r}_{n-1} \le 0,\ \bar{r}_n > 0 \\ (1+\bar{r}_{n-1})x_{n-1}, \text{if } \bar{r}_{n-1} > 0,\ \bar{r}_n \le 0 \end{cases}$$

$$\Rightarrow f_{n-1}(x_{n-1}) = \prod_{\bar{r} \in \bar{R}_{n-1}^{+}} (1+\bar{r})x_{n-1}$$

Therefore, the theorem is also valid for $i = n-1$.

3.5 Stage *n* – 2 Computations

$$x_{i+1} = x_i + r_{k_i} y_i;\ i \in \{1, 2, \ldots, n\}$$

$$\Rightarrow f_{n-2}(x_{n-2}) = \max_{0 \le y_{n-2} \le x_{n-2}} \left\{ \sum_{k_{n-2}=1}^{m_{n-2}} p_{k_{n-2}} f_{n-1}(x_{n-1}) \right\}$$

$$= \max_{0 \le y_{n-2} \le x_{n-2}} \left\{ \sum_{k_{n-2}=1}^{m_{n-2}} p_{k_{n-2}} \begin{cases} x_{n-1}, \text{ if } \bar{r}_{n-1}, \bar{r}_n \le 0 \\ (1+\bar{r}_n)(1+\bar{r}_{n-1})x_{n-1}, \text{if } \bar{r}_{n-1}, \bar{r}_n > 0 \\ (1+\bar{r}_n)x_{n-1}, \text{if } \bar{r}_{n-1} \le 0,\ \bar{r}_n > 0 \\ (1+\bar{r}_{n-1})x_{n-1}, \text{if } \bar{r}_{n-1} > 0,\ \bar{r}_n > 0 \end{cases} \right\}$$

$$= \max_{0 \le y_{n-2} \le x_{n-2}} \left\{ \begin{cases} \sum_{k_{n-2}=1}^{m_{n-2}} p_{k_{n-2}} \left(x_{n-2} + r_{k_{n-2}} y_{n-2}\right), \text{ if } \bar{r}_{n-1}, \bar{r}_n \le 0 \\ \sum_{k_{n-2}=1}^{m_{n-2}} p_{k_{n-2}} (1+\bar{r}_n)(1+\bar{r}_{n-1})\left(x_{n-2} + r_{k_{n-2}} y_{n-2}\right), \text{if } \bar{r}_{n-1}, \bar{r}_n > 0 \\ \sum_{k_{n-2}=1}^{m_{n-2}} p_{k_{n-2}} (1+\bar{r}_n)\left(x_{n-2} + r_{k_{n-2}} y_{n-2}\right), \text{if } \bar{r}_{n-1} \le 0,\ \bar{r}_n > 0 \\ \sum_{k_{n-2}=1}^{m_{n-2}} p_{k_{n-2}} (1+\bar{r}_{n-1})\left(x_{n-2} + r_{k_{n-2}} y_{n-2}\right), \text{if } \bar{r}_{n-1} > 0,\ \bar{r}_n \le 0 \end{cases} \right\}$$

$$= \begin{cases} x_{n-2} + \bar{r}_{n-2} y_{n-2}, \text{ if } \bar{r}_{n-1}, \bar{r}_n \le 0 \\ (1+\bar{r}_n)(1+\bar{r}_{n-1})\left(x_{n-2} + \bar{r}_{n-2} y_{n-2}\right), \text{if } \bar{r}_{n-1}, \bar{r}_n > 0 \\ (1+\bar{r}_n)\left(x_{n-2} + \bar{r}_{n-2} y_{n-2}\right), \text{if } \bar{r}_{n-1} \le 0,\ \bar{r}_n > 0 \\ (1+\bar{r}_{n-1})\left(x_{n-2} + \bar{r}_{n-2} y_{n-2}\right), \text{if } \bar{r}_{n-1} > 0,\ \bar{r}_n \le 0 \end{cases}$$

$$\Rightarrow y^*_{n-2} = \begin{cases} 0, \text{ if } \bar{r}_{n-2}, \bar{r}_{n-1}, \bar{r}_n \le 0 \\ x_{n-2}, \text{if } \bar{r}_{n-2} > 0, \bar{r}_{n-1}, \bar{r}_n \le 0 \\ x_{n-2}, \text{if } \bar{r}_{n-2}, \bar{r}_{n-1}, \bar{r}_n > 0 \\ 0, \text{ if } \bar{r}_{n-2} \le 0, \bar{r}_{n-1}, \bar{r}_n > 0 \\ 0, \text{ if } \bar{r}_{n-1} \le 0, \bar{r}_{n-1} \le 0, \bar{r}_n > 0 \\ x_{n-2}, \text{if } \bar{r}_{n-2} > 0, \bar{r}_{n-1} \le 0, \bar{r}_n > 0 \\ 0, \text{ if } \bar{r}_{n-2} \le 0, \bar{r}_{n-1} > 0, \bar{r}_n \le 0 \\ x_{n-2}, \text{if } \bar{r}_{n-2} > 0, \bar{r}_{n-1} > 0, \bar{r}_n \le 0 \end{cases}$$

$$\Rightarrow y^*_{n-2} = \begin{cases} 0, \text{if } \bar{r}_{n-2} \le 0 \\ x_{n-2}, \text{if } \bar{r}_{n-2} > 0 \end{cases}$$
$$= x_{n-2} \operatorname{sgn}\left(\max\{\bar{r}_{n-2}, 0\}\right)$$

$$\Rightarrow f_{n-2}(x_{n-2}) = \begin{cases} x_{n-2}, \text{ if } \bar{r}_{n-2}, \bar{r}_{n-1}, \bar{r}_n \le 0 \\ (1+r_{n-2})x_{n-2}, \text{if } \bar{r}_{n-2} > 0, \bar{r}_{n-1}, \bar{r}_n \le 0 \\ (1+\bar{r}_n)(1+\bar{r}_{n-1})x_{n-2}, \text{if } \bar{r}_{n-2} \le 0, \bar{r}_{n-1}, \bar{r}_n > 0 \\ (1+\bar{r}_n)(1+\bar{r}_{n-1})(1+\bar{r}_{n-2})x_{n-2}, \text{ if } \bar{r}_{n-2}, \bar{r}_{n-1}, \bar{r}_n > 0 \\ (1+\bar{r}_n)x_{n-2}, \text{ if } \bar{r}_{n-2} \le 0, \bar{r}_{n-1} \le 0, \bar{r}_n > 0 \\ (1+\bar{r}_n)(1+\bar{r}_{n-2})x_{n-2}, \text{ if } \bar{r}_{n-2} > 0, \bar{r}_{n-1} \le 0, \bar{r}_n > 0 \\ (1+\bar{r}_{n-1})x_{n-2}, \text{if } \bar{r}_{n-2} \le 0, \bar{r}_{n-1} > 0, \bar{r}_n \le 0 \\ (1+\bar{r}_{n-1})(1+\bar{r}_{n-2})x_{n-2}, \text{if } \bar{r}_{n-2} > 0, \bar{r}_{n-1} > 0, \bar{r}_n \le 0 \end{cases}$$

$$\Rightarrow f_{n-2}(x_{n-2}) = \begin{cases} x_{n-2}, \text{ if } \bar{R}^+_{n-2} = \varnothing \\ \prod_{\bar{r} \in \bar{R}^+_{n-2}} (1+\bar{r})x_{n-2}, \text{elsewhere} \end{cases}$$

Therefore the theorem is also valid for $i = n-2.$

3.6 Stages *n* –3 to1 Computations

Assume that the theorem is valid for $j \le i \le n-3,$ for some positive integer $j > 1$. Then in particular,

$$y^*_j = x_j \operatorname{sgn}\left(\max\{\bar{r}_j, 0\}\right)$$
$$f_j(x_j) = \begin{cases} x_j, \text{ if } \bar{R}_j = \bar{R}^-_j \\ \prod_{\bar{r} \in \bar{R}^+} (1+\bar{r})x_j, \text{ if } \bar{R}^+ \ne \varnothing \end{cases}$$

$f_{j-1}(x_{j-1})$ can now be effectively examined.

$$f_{j-1}(x_{j-1}) = \max_{0\le y_{j-1}\le x_{j-1}} \left(\sum_{k_{j-1}=1}^{m_{j-1}} p_{k_{j-1}} f_j(x_j) \right) = \max_{0\le y_{j-1}\le x_{j-1}} \left\{ \sum_{k_{j-1}=1}^{m_{j-1}} p_{k_{j-1}} \begin{cases} x_j, \text{ if } \bar{R}_j^+ = \varnothing \\ \prod_{\bar{r}\in\bar{R}_j^+} (1+\bar{r}) x_j, \text{otherwise} \end{cases} \right\}$$

$$= \max_{0\le y_{j-1}\le x_{j-1}} \begin{cases} \sum_{k_{j-1}=1}^{m_{j-1}} p_{k_{j-1}} \left(x_{j-1} + r_{k_{j-1}} y_{j-1} \right), \text{if } \bar{R}_j^- = \bar{R}_j \\ \sum_{k_{j-1}=1}^{m_{j-1}} p_{k_{j-1}} \prod_{\bar{r}\in\bar{R}_j^+} (1+\bar{r}) \left(x_{j-1} + r_{j-1} y_{j-1} \right), \text{if } \bar{R}_j^+ \ne \varnothing \end{cases}$$

$$= \max_{0\le y_{j-1}\le x_{j-1}} \begin{cases} x_{j-1} + \bar{r}_{j-1} y_{j-1} \ \text{ if } \bar{R}_j^- = \bar{R}_j \left(\text{i.e.} \bar{R}_j^+ = \varnothing \right) \\ \prod_{\bar{r}\in\bar{R}_j^+} (1+\bar{r}) \left(x_{j-1} + \bar{r}_{j-1} y_{j-1} \right), \text{if } \bar{R}_j^+ \ne \varnothing \end{cases}$$

$$\Rightarrow y^*_{j-1} = x_{j-1} \operatorname{sgn}\left(\max\left\{ \bar{r}_{j-1}, 0 \right\} \right)$$

$$\Rightarrow f_{j-1}(x_{j-1}) = \begin{cases} x_{j-1}, \text{ if } \bar{r}_{j-1} \le 0, \text{ and } \bar{R}_j^- = \bar{R}_j \\ (1+\bar{r}) x_{j-1}, \text{ if } \bar{r}_{j-1} > 0, \text{ and } \bar{R}_j^- = \bar{R}_j \\ \prod_{\bar{r}\in\bar{R}_j^+} (1+\bar{r}) x_{j-1}, \text{ if } \bar{r}_{j-1} \le 0, \text{and } \bar{R}_j^+ \ne \varnothing \\ \prod_{\bar{r}\in\bar{R}_j^+} (1+\bar{r}) \left(1+\bar{r}_{j-1} \right) x_{j-1}, \text{ if } \bar{r}_{j-1} > 0, \text{and } \bar{R}_j^+ \ne \varnothing \end{cases}$$

$$\Rightarrow f_{j-1}(x_{j-1}) = \begin{cases} x_{j-1}, \text{ if } \bar{R}_{j-1}^- = \bar{R}_{j-1} \left(\text{i.e. if } \bar{R}_{j-1}^+ = \varnothing \right) \\ \prod_{\bar{r}\in\bar{R}_{j-1}^+} (1+\bar{r}) x_{j-1}, \text{ if } \bar{R}_{j-1}^+ \ne \varnothing \end{cases}$$

Therefore, the theorem is valid for $i = j-1$ and hence valid for all $1 \le i \le n.$ This completes the proof.

3.7 Corollary to Theorem 2

If $m_i = m,$ and $k_i = k, \forall\ i \in \{1, 2, \ldots, n\}$, then

$$f_i(x_i) = \begin{cases} x_i, \text{ if } \bar{r} \le 0 \\ \prod_{p=i}^{n} (1+\bar{r}) x_i, \text{ if } \bar{r} > 0 \end{cases}$$

$$y_i^* = \begin{cases} 0, & \text{if } \bar{r} \le 0 \\ x_i, & \text{if } \bar{r} > 0 \end{cases}$$

This is consistent with theorem 1.

Proof of the corollary

$m_i = m$ and $k_i = k, \forall i \in \{1, 2, \cdots, n\} \Rightarrow p_{k_i} = p_k$

$\Rightarrow \bar{r}_i = \sum_{k_i=1}^{m_i} p_{k_i} r_{k_i} = \sum_{k=1}^{m} p_k r_k = \bar{r} \Rightarrow \bar{R}_i^+ = \{\bar{r} > 0\}, \bar{R}_i^- = \{\bar{r} \le 0\} \Rightarrow \bar{R}_i^+ = \{\bar{r}_p > 0, \forall p \in \{i, \cdots, n\}\}.$

$\bar{R}_i^- = \{\bar{r}_p \le 0, \forall p \in \{i, \cdots, n\}\} \Rightarrow |\bar{R}_i^+| = n + 1 - i$ and $|\bar{R}_i^-| = 0$ or $|\bar{R}_i^-| = n + 1 - i$ and $|\bar{R}_i^+| = 0,$

where $|S|$ denotes the cardinality of an arbitrary set S; $\bar{R}_i^- \ne \varnothing \Leftrightarrow \bar{R}_i^+ = \varnothing \Leftrightarrow \bar{R}_i^- = \bar{R}_i$;

$\bar{R}_i^+ \ne \varnothing \Leftrightarrow \bar{R}_i^- = \varnothing \Leftrightarrow \bar{R}_i^+ = \bar{R}_i \Rightarrow \prod_{\bar{r} \in \bar{R}_i^+} (1+\bar{r}) x_i = (1+\bar{r})^{n+1-i} x_i$; "if $\bar{R}_i^+ \ne \varnothing$" $\Leftrightarrow$ " if $\bar{r} > 0$"

"if $\bar{R}_i^+ = \varnothing$" $\Leftrightarrow$ " if $\bar{r} \le 0$". Clearly, $(1+\bar{r})^{n+1-i} x_i = \prod_{p=i}^{n} (1+\bar{r}) x_i$

$$\Rightarrow f_i(x_i) = \begin{cases} x_i; \text{ if } \bar{r} \le 0 \\ \prod_{p=i}^{n} (1+\bar{r}) x_i, \text{ if } \bar{r} > 0 \end{cases}$$

$$y_i^* = \begin{cases} 0, & \text{if } \bar{r} \le 0 \\ x_i, & \text{if } \bar{r} > 0. \end{cases}$$

This concludes the proof of the corollary.

Problem 1

Obtain the optimal investment strategy given the following pertinent balanced data.

Table 1: Market returns and associated probabilities for years 1-4

***C* = $10,000**						
Year i	**r1**	**r2**	**r3**	**p1**	**p2**	**p3**
1	2	1	0.5	0.1	0.4	0.5
2	1	0	-1	0.4	0.4	0.5
3	4	-1	-1	0.2	0.4	0.4
4	0.8	0.4	0.2	0.6	0.2	0.2

Solution

$n = 4, m_i = m = 3, \bar{r}_1 = 0.85, \bar{r}_2 = -0.1, \bar{r}_3 = 0,$ and $\bar{r}_4 = 0.6.$

It follows from theorem 2 that

$$y_4^* = x_4,\ y_3^* = 0,\ \ y_2^* = 0, y_1^* = x_1 = \$10,000.00;$$
$$f_4(x_4) = \left(1+\bar{r}_4\right)x_4 = 1.6x_4,\ f_3(x_3) = \left(1+\bar{r}_4\right)x_3 = 1.6x_3,\ f_2(x_2) = \left(1+\bar{r}_4\right)x_2 = 1.6x_2,$$
$$f_1(x_1) = \left(1+\bar{r}_1\right)\left(1+\bar{r}_4\right)x_1 = 1.85(1.60)x_1 = 2.96x_1 = \$29,600.00.$$

3.8 Optimal Investment Policy

The optimal solution calls for investing all available funds at the beginning of years 1 and 4 and none at the beginning of years 2 and 3. The accumulated funds at the end of the 4th year is $f_1(x_1) = \$29,600.00$.

Problem 2: Obtain the optimal investment policy given the following pertinent dynamic data.

Table 2: Market returns and associated probabilities for years 1-10

Year i	Market Returns					Associated Probabilities				
1	2	1	0.5			0.1	0.3	0.2	0.25	0.15
2	1	0	-1	0.4		0.4	0.2	0.1	0.15	0.15
3	4	-1	-1			0.02	0.14	0.24	0.4	0.2
4	0.8	0.4	0.2	0.1	0	0.3	0.2	0.2	0.2	0.1
5	0.7	0.5	1	-1		0.1	0.2	0.2	0.2	0.3
6	3	-1				0.05	0.2	0.2	0.2	0.35
7	0.9	0.7	0.2	-1	1	0.15	0.2	0.2	0.4	0.05
8	1.5	1	0.6			0.3	0.2	0.2	0.2	0.1
9	-1	0.5	0.4			0.6	0.2	0.1	0.1	0
10	0	1	1.5			0.35	0.2	0.2	0.2	0.05

Solution:

Table 3: Table of Computed relevant values

Year I	**5**	**6**	**7**	**8**	**9**	**10**
$\bar{r}_i$	0.17	-0.05	-0.035	0.77	-0.46	0.50
y_i^*	x_5	0	0	x_8	0	x_{10}
$f_i(x_i)$	$\prod_{i \in \{10,8,5\}} (1+\bar{r}_i)x_5$	$(1+\bar{r}_{10})(1+\bar{r}_8)x_6$	$(1+\bar{r}_{10})(1+\bar{r}_8)x_7$	$(1+\bar{r}_{10})(1+\bar{r}_8)x_8$	$(1+\bar{r}_{10})x_9$	$(1+\bar{r}_{10})x_{10}$
=	$3.10635x_5$	$2.655x_6$	$2.655x_7$	$2.655x_8$	$1.5x_9$	$1.5x_{10}$
Year I	**1**		**2**	**3**		**4**

$\bar{r}_i$	0.6	0.36	-0.30	0.38
y_i^*	$x_1 = 10,000$	x_2	0	x_4
$f_i(x_i)$	$\prod_{i \in \{10,8,5,4,2,1\}} (1+\bar{r}_i)x_1$	$\prod_{i \in \{10,8,5,4,2\}} (1+\bar{r}_i)x_2$	$\prod_{i \in \{10,8,5,4\}} (1+\bar{r}_i)x_3$	$\prod_{i \in \{10,8,5,4\}} (1+\bar{r}_i)x_3$
=	$9.327996288x_1$	$5.82999768x_2$	$4.286763x_3$	$4.286763x_4$

Therefore, the optimal objective value is 93279.96288 $\approx \$93,279.96$

3.9 Optimal Investment Policy Prescription

Invest all available funds at the beginning of years 1, 2, 4, 5, 8 and10 and none at all, at the beginning of years 3, 6, 7 and 9. The expected accumulated funds at the end of the 10 years $= f_1(x_1) = \$93,279.96$.

4. Conclusion

This article furnished the proof of the optimal investment strategy and optimal return for a class of investment problems with stationary returns under uncertainty, using the principle of mathematical induction, as suggested by [2]. The work went much further to extend the results to a much larger class of problems of the complex dynamic class, where the probabilities and market returns vary from period to period; that is, for arbitrary number of different market conditions for each period with different probabilities and corresponding returns. The unprecedented extension and optimal policy prescriptions are encapsulated in theorem 2, which is the ‘icing on the cake’. The results circumvent the inherent tedious and prohibitive computations associated with dynamic programming formulations and may be optimally appropriated for sensitivity analyses on such models. Finally the article provided two illustrative problems for the optimal policy prescription, with respect to the dynamic class.

References

[1] Winston, L.W., *Operations research*: *Applications and Algorithm*, (Duxbury Press, Boston, 2004) pp 967.

[2] Taha, H.A., *Operations research*: *an introduction, Seventh Edition* (Prentice-Hall of India, New Delhi, 2006) pp. 404.

[3] Verma, A.P., *Operations research*: (S.K. Kataria & Sons. New Delhi, 2010).

[4] Gupta, P.K. and Hira, D.S., *Problems in operations research; principles and solutions*, (S. Chand & Company Ltd. New Delhi, 2012).

[5] Wagner, H. M., *Principles of operations research, with applications to management decisions*, *2nd Edition*, (Prentice-Hall of India, New Delhi, 2000).

[6] Taha H.A., *Operations research: an introduction*, *Eight Edition*, (Prentice-Hall of India, New Delhi, 2007).

[7] Pan, Y. and Theodorou, E., *Probabilistic differential dynamic programming*, (In Advances in Neural Information Processing Systems (NIPS) 2014), pp. 1907-1915.