Weakly Compatible Maps For Three Self Maps In Complex Valued Metric Spaces

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Abstract. In this paper, we prove some common fixed point theorems for three self maps in complex valued metric space. Also, we give suitable example in support of our results.

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1. Introduction

In 2011, Azam et. al. [1] introduced the notion of complex valued metric space which is a generalization of the classical metric space. They established some fixed point results for mappings satisfying a rational inequality. The idea of complex valued metric spaces can be expolited to define complex valued normed spaces and complex valued Hilbert spaces; additionally, it offers numerous research activities in mathematical analysis.

A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first co-ordinate is called Re(z) and second coordinate is called Im(z). Thus a complex-valued metric d is a function from a set $X \times X$ into $\mathbb{C}$, where X is a nonempty set and $\mathbb{C}$ is the set of complex numbers.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$, that is $z_1 \precsim z_2$, if one of the following holds:

(C1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;

(C4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3), and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied.

Remark 1.1. We note that the following statements hold:

(i) $a, b \in \mathbb{R}$ and $a \leq b \Rightarrow az \precsim bz\ \forall z \in \mathbb{C}$.

(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$,

(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 1.2. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(i) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1.3. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$d(z_1, z_2) = 2\iota\,|z_1 - z_2|$, for all $z_1, z_2 \in X$.

Then (X, d) is a complex valued metric space.

Definition 1.4. Let (X, d) be a complex valued metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $k \in \mathbb{N}$ such that for all $n > k$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\{x_n\} \to x$ as $n \to \infty$ or $\lim_{n \to \infty} x_n = x$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $k \in \mathbb{N}$ such that for all $n > k$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex valued metric space.

Lemma 1.5. Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then {xn} converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 1.6. Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

In 1996, Jungck [2] introduced the notion of weakly compatible maps as follows:

Definition 1.7. Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

2. Main Results

Theorem 2.1. Let S, T and f be three self maps of a complex valued metric space (X, d) satisfying the following:

(2.1) $SX \cup TX \subseteq fX$,

(2.2) $d(Sx, Ty) \precsim h\, d(fx, fy)$, for all x, y in X,

where $0 \leq h < 1$,

(2.3) fX is complete subspace of X.

Then S, T and f have a unique coincidence point. Moreover, if (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Proof. Let $x_0 \in X$. Choose a point x_1 in X such that $fx_1 = Sx_0$. This can be done, since $SX \subseteq fX$. Similarly, choose a point x_2 in X such that $fx_2 = Tx_1$. Continuing this process and having chosen x_n in X, we obtain x_{n+1} in X such that

$$fx_{2k+1} = Sx_{2k},\ fx_{2k+2} = Tx_{2k+1},\ k = 0, 1, 2, \ldots.$$

From (2.2), we have

$$d(fx_{2k+1}, fx_{2k+2}) = d(Sx_{2k}, Tx_{2k+1}) \precsim h\, d(fx_{2k}, fx_{2k+1}).$$

Similarly,

$$d(fx_{2k+2}, fx_{2k+3}) \precsim h\, d(fx_{2k+1}, fx_{2k+2}).$$

Now, by induction, we obtain for each k = 0, 1, 2, . . .,

$$d(fx_{2k+2}, fx_{2k+3}) \precsim h\, d(fx_0, fx_1).$$

Let $y_n = fx_n$, n = 0, 1, 2, ...

Now, for all n, we have

$$d(y_{n+1}, y_{n+2}) \precsim h\, d(y_n, y_{n+1}) \precsim h^2\, d(y_{n-1}, y_n) \precsim \ldots \precsim h^{n+1}\, d(y_0, y_1).$$

Now, for any m > n,

$$d(y_m, y_n) \precsim d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m)$$
$$\precsim [h^n + h^{n+1} + \ldots + h^{m-1}]\, d(y_0, y_1)$$
$$\precsim \frac{h^n}{1-h}\, d(y_0, y_1).$$

Therefore, we have

$|d(y_m, y_n)| \leq \frac{k^n}{1-k} |d(y_0, y_1)|$.

Hence,

$\lim_{n\to\infty} |d(y_m, y_n)| = 0$,

which implies that $\{y_n\}$ is a Cauchy sequence. Since fX is complete, there exists u, v in X such that $y_n \to v = fu$.

Choose a natural number N such that

$$d(y_n, v) \prec \frac{c}{2}, \text{ for all } n \geq N.$$

Hence, for all $n \geq N$, using triangle inequality and (2.2), we have

$$\begin{aligned} d(fu, Su) &\precsim d(fu, y_{2n+2}) + d(y_{2n+2}, Su) \\ &= d(v, y_{2n+2}) + d(Tx_{2n+1}, Su) \\ &\precsim d(v, y_{2n+2}) + h\, d(fx_{2n+1}, fu) \\ &\precsim d(v, y_{2n+2}) + h\, d(y_{2n+1}, v) \prec \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

Thus, $d(fu, Su) \precsim \frac{c}{m}$ for all $m \geq 1$, that is,

$|d(fu, Su)| \leq \frac{c}{m}$.

But, since m was arbitrary, so

$|d(fu, Su)| = 0$, implies that, fu = Su.

Similarly, by using

$d(fu, Tu) \precsim d(fu, y_{2n+1}) + d(y_{2n+1}, Tu)$,

one can show that fu = Tu, it implies that, v is a common point of coincidence of S, T and f, that is, v = fu = Su = Tu.

Now, we show that f, S and T have a unique point of coincidence. For this, assume that there exists another point w in X such that w = fz = Sz = Tz for some z in X.

From (2.2), we have

$$\begin{aligned} d(v, w) &= d(Su, Tz) \\ &\precsim h\, d(fu, fz) = h\, d(v, w), \text{ implies that, } v = w. \end{aligned}$$

Now, since (S, f) and (T, f) are weakly compatible, we have

Sv = Sfu = fSu = fv and Tv = Tfu = fTu = fv.

It implies that Sv = Tv = fv = t (say). Then w is a point of coincidence of S, T and f.

Therefore, v = t, by uniqueness.

Hence v is a common fixed point of S, T and f.

Theorem 2.2. Let S, T and f be three self maps of a complex valued metric space (X, d) satisfying (2.1), (2.3) and the following:

(2.4) $d(Sx, Ty) \precsim h\,[d(fx, Sx) + d(fy, Ty)]$, for all x, y in X,

where $0 \le h < \frac{1}{2}$.

Then S, T and f have a unique coincidence point. Moreover, if (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Proof. Let $x_0 \in X$. Define a sequence of points in X, as in Theorem 2.1, given by the rule:

$fx_{2k+1} = Sx_{2k}$, $fx_{2k+2} = Tx_{2k+1}$, k = 0, 1, 2, ...

From (2.4), we have

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &\precsim h\,[d(fx_{2k}, Sx_{2k}) + d(fx_{2k+1}, Tx_{2k+1})] \\ &= h\,[d(fx_{2k}, fx_{2k+1}) + d(fx_{2k+1}, fx_{2k+2})], \text{ that is,} \end{aligned}$$

$$d(fx_{2k+1}, fx_{2k+2}) \precsim \frac{h}{1-h}\, d(fx_{2k}, fx_{2k+1}).$$

Similarly, it can be shown that

$$\begin{aligned} d(fx_{2k+2}, fx_{2k+3}) &\precsim \frac{h}{1-h}\, d(fx_{2k+1}, fx_{2k+2}) \\ &= p\, d(fx_{2k+1}, fx_{2k+2}),\ p = \frac{h}{1-h} < 1. \end{aligned}$$

Now, by induction, we obtain for each k = 0, 1, 2, . . . ,

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &\precsim p\, d(fx_{2k}, fx_{2k+1}) \\ &\precsim p^2\, d(fx_{2k-1}, fx_{2k}) \precsim \ldots \precsim p^{2k+1}\, d(fx_0, fx_1). \end{aligned}$$

Let $y_n = fx_n$, n= 0, 1, 2, ...

Now, for all n, we have

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\precsim p\, d(y_n, y_{n+1}) \\ &\precsim p^2\, d(y_{n-1}, y_n) \precsim \ldots \precsim p^{n+1}\, d(y_0, y_1). \end{aligned}$$

Now, for any m > n,

$$d(y_m, y_n) \precsim d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m)$$

$\precsim [p^n + p^{n+1} + \ldots + p^{m-1}]\, d(y_0, y_1)$

$\precsim \frac{p^n}{1-p}\, d(y_0, y_1).$

Therefore, we have

$|d(y_m, y_n)| \leq \frac{p^n}{1-p}\, |d(y_0, y_1)|.$

Hence,

$\lim_{n\to\infty} |d(y_m, y_n)| = 0,$

which implies that $\{y_n\}$ is a Cauchy sequence.

Since fX is complete, there exists u, v in X such that $y_n \to v = fu$.

Choose a natural number N such that

$d(y_{n+1}, y_n) \prec \frac{c(1-h)}{2h}$ and $d(y_{n+1}, v) \prec \frac{c(1-h)}{2}$, for all $n \geq N$.

Hence, for all $n \geq N$, using triangle inequality and (2.4), we have

$d(fu, Su) \precsim d(fu, y_{2n+2}) + d(y_{2n+2}, Su)$

$= d(v, y_{2n+2}) + d(Tx_{2n+1}, Su)$

$\precsim d(v, y_{2n+2}) + h\, [d(fu, Su) + d(fx_{2n+1}, Tx_{2n+1})]$, that is,

$d(fu, Su) \precsim \frac{1}{1-h}\, d(v, y_{2n+2}) + \frac{h}{1-h}\, d(y_{2n+1}, y_{2n+2})$

$\prec \frac{c}{2} + \frac{c}{2} = c.$

Thus, $d(fu, Su) \precsim \frac{c}{m}$ for all $m \geq 1$, that is,

$|d(fu, Su)| \leq \frac{c}{m}.$

But, since m was arbitrary, so

$|d(fu, Su)| = 0$, implies that, $fu = Su$.

Similarly, by using

$d(fu, Tu) \precsim d(fu, y_{2n+1}) + d(y_{2n+1}, Tu),$

one can show that fu = Tu, it implies that, v is a common point of coincidence of S, T and f, that is, v = fu = Su = Tu.

Now, we show that f, S and T have a unique point of coincidence. For this, assume that there exists another point w in X such that w = fz = Sz = Tz for some z in X.

From (2.4), we have

$d(v, w) = d(Su, Tz)$

$\precsim h\,[d(fu, Su) + d(fz, Tz)]$

$= h\,[d(v, v) + d(w, w)] = 0$, that is,

$|d(v, w)| \leq 0$, implies that, $v = w$.

Now, since (S, f) and (T, f) are weakly compatible, we have

$Sv = Sfu = fSu = fv$ and $Tv = Tfu = fTu = fv$.

It implies that $Sv = Tv = fv = t$ (say). Then w is a point of coincidence of S, T and f.

Therefore, $v = t$, by uniqueness.

Hence v is a common fixed point of S, T and f.

Theorem 2.3. Let S, T and f be three self maps of a complex valued metric space (X, d) satisfying (2.1), (2.3) and the following:

(2.5) $d(Sx, Ty) \precsim h\,[d(fy, Sx) + d(fx, Ty)]$, for all x, y in X,

where $0 \leq h < \frac{1}{2}$.

Then S, T and f have a unique coincidence point. Moreover, if (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Proof. Let $x_0 \in X$. Define a sequence of points in X, as in Theorem 2.1, given by the rule:

$fx_{2k+1} = Sx_{2k}$, $fx_{2k+2} = Tx_{2k+1}$, $k = 0, 1, 2, \ldots$.

From (2.5), we have

$d(fx_{2k+1}, fx_{2k+2}) = d(Sx_{2k}, Tx_{2k+1})$

$\precsim h\,[d(fx_{2k+1}, Sx_{2k}) + d(fx_{2k}, Tx_{2k+1})]$

$= h\,[d(fx_{2k+1}, fx_{2k+1}) + d(fx_{2k}, fx_{2k+2})]$

$= h\, d(fx_{2k}, fx_{2k+2})$

$\precsim h\,[d(fx_{2k}, fx_{2k+1}) + d(fx_{2k+1}, Tx_{2k+2})]$, that is,

$d(fx_{2k+1}, fx_{2k+2}) \precsim \frac{h}{1-h}\, d(fx_{2k}, fx_{2k+1})$.

Similarly, it can be shown that

$d(fx_{2k+2}, fx_{2k+3}) \precsim \frac{h}{1-h}\, d(fx_{2k+1}, fx_{2k+2})$

$= p\, d(fx_{2k+1}, fx_{2k+2})$, $p = \frac{h}{1-h} < 1$.

Now, by induction, we obtain for each $k = 0, 1, 2, \ldots$,

$d(fx_{2k+1}, fx_{2k+2}) \precsim p\, d(fx_{2k}, fx_{2k+1})$

$\precsim p^2\, d(fx_{2k-1}, fx_{2k}) \precsim \ldots \precsim p^{2k+1}\, d(fx_0, fx_1)$.

Let $y_n = fx_n$, $n = 0, 1, 2, \ldots$.

Now, for all n, we have

$d(y_{n+1}, y_{n+2}) \precsim p\, d(y_n, y_{n+1})$

$\precsim p^2\, d(y_{n-1}, y_n) \precsim \ldots \precsim p^{n+1}\, d(y_0, y_1)$.

Now, for any $m > n$,

$d(y_m, y_n) \precsim d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m)$

$\precsim [p^n + p^{n+1} + \ldots + p^{m-1}]\, d(y_0, y_1)$

$\precsim \frac{p^n}{1-p}\, d(y_0, y_1)$.

Therefore, we have

$|d(y_m, y_n)| \leq \frac{p^n}{1-p} |d(y_0, y_1)|$.

Hence,

$\lim_{n\to\infty} |d(y_m, y_n)| = 0$,

which implies that $\{y_n\}$ is a Cauchy sequence.

Since fX is complete, there exists u, v in X such that $y_n \to v = fu$.

Choose a natural number N such that

$d(y_{n+1}, y_n) \prec \frac{c(1-h)}{3}$, for all $n \geq N$.

Hence, for all $n \geq N$, using triangle inequality and (2.5), we have

$d(fu, Su) \precsim d(fu, y_{2n+1}) + d(y_{2n+1}, Su)$

$= d(v, y_{2n+1}) + d(Tx_{2n+1}, Su)$

$\precsim d(v, y_{2n+1}) + h\,[d(fu, Tx_{2n+1}) + d(fx_{2n+1}, Su)]$

$= d(v, y_{2n+1}) + h\,[d(v, y_{2n+2}) + d(y_{2n+1}, Su)]$, that is,

$d(fu, Su) \precsim d(v, y_{2n+1}) + h\,[d(v, y_{2n+2}) + d(y_{2n+1}, v) + d(v, Su)]$, implies that,

$d(fu, Su) \precsim \frac{1}{1-h}\, d(v, y_{2n+1}) + \frac{h}{1-h}\,[d(v, y_{2n+2}) + d(y_{2n+1}, v)]$

$\prec \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c$.

Thus, we have

d(fu, Su) $\precsim \frac{c}{m}$, for all m ≥ 1, that is,

$|d(fu, Su)| \leq \frac{c}{m}$.

But, since m was arbitrary, so

$|d(fu, Su)|$ = 0, implies that, fu = Su.

Similarly, by using

d(fu, Tu) $\precsim$ d(fu, y_{2n+1}) + d(y_{2n+1}, Tu),

one can show that fu = Tu, it implies that, v is a common point of coincidence of S, T and f, that is, v = fu = Su = Tu.

Now, we show that f, S and T have a unique point of coincidence. For this, assume that there exists another point w in X such that w = fz = Sz = Tz for some z in X.

From (2.5), we have

d(v, w) = d(Su, Tz)

$\precsim$ h [d(fz, Su) + d(fu, Tz)]

= h [d(w, v) + d(v, w)] = 2h d(v, w), that is,

$|d(v, w)| \leq 2h\, |d(v, w)|$, implies that, v = w.

Now, since (S, f) and (T, f) are weakly compatible, we have

Sv = Sfu = fSu = fv and Tv = Tfu = fTu = fv.

It implies that Sv = Tv = fv = t (say). Then w is a point of coincidence of S, T and f.

Therefore, v = t, by uniqueness.

Hence v is a common fixed point of S, T and f.

Example 2.4. Let X = [0, 1] and let d : X × X → ℂ by d(x, y) = $i\, |x - y|$, for all x, y ∈ X .

Then (X, d) is a complex valued metric space.

Define the functions S, T, f : X → X by

Sx = $\frac{x}{4}$ = Tx and fx = $\frac{x}{2}$.

Clearly SX ∪ TX = [0, $\frac{1}{4}$] ⊆ [0, $\frac{1}{2}$] = fX.

Also (S, f) and (T, f) are weakly compatible.

Now,

d(Sx, Ty) = $\frac{i}{4}\, |x - y|$, d(fx, fy) = $\frac{i}{2}\, |x - y|$.

Clearly, for h = $\frac{2}{3} < 1$,

d(Sx, Ty) $\precsim$ h d(fx, fy).

Also 0 is the common fixed point of S, T and f.

Hence all the conditions of Theorem 2.1 are satisfied.

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