A PROMINENT OBSERVATION ON NUMBER THEORETIC FUNCTION $\tau(n)$

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ABSTRACT

The aim of this paper is observe the certain results of number theoretic function $\tau(n)$ *for all positive finite integers* $n \geq 1$. *If* n *is any finite positive integer, then* $\tau^k(n) = 2$ *for all positive integers* $k \geq N$, *where* $1 \leq N \leq k$.

Key Words: Arithmetic function, Number Theoretic Function, Prime Numbers.

Introduction: This paper focused on certain results of number theoretic function $\tau(n)$. A number theoretic function or simply arithmetic function is any function whose domain of definition is the set of positive integers [Burton, 2008]. One such function the τ, is herein defined-

Definition (1): Let n be a given positive integer. Then $\tau(n)$ is the number of positive divisors of n including the divisors 1 and n.

Examples:

(1) The divisor of 1 is only 1. Hence $\tau(1) = 1$

(2) The divisors of 12 are 1, 2, 3, 4, 6 and 12. Hence $\tau(n) = 6$.

(3) The divisors of 17 are 1 and 17 only. Hence $\tau(n) = 2$.

Theorem (1): If n is a prime number then $\tau(n) = 2$.

Proof: Since we know that every prime number has only two divisors 1 and itself. Therefore by the definition of $\tau(n)$, $\tau(n) = 2$.

Theorem (2): Let m *and* n are two relatively prime numbers, then $\tau(m, n) = \tau(m)\tau(n)$.

Proof: Since we know that $gcd(m, n) = 1$ it follows that m and n has different divisors to each other's. Hence

$\tau(m, n) = \tau(m)\tau(n)$.

Corollary (1): Let $n = p\,q$, where p and q are two prime numbers such that $p \neq q$. Then $\tau(n) = 4$.

Proof: Since we know that p and q are two prime numbers such that $p \neq q$ therefore $gcd(p, q) = 1$. Hence $\tau(n) = \tau(p, q) = \tau(p)\tau(q) = 2.2 = 4$ from theorem (1) and theorem (2).

Theorem (3): Let $n = p^{k_1}$, then $\tau(n) = (k_1 + 1)$.

Proof: Since the divisors of n are 1 and p^r, where $r = 1,2,3, \dots\dots\dots\dots, k_1$ i. e. the total divisors of n are

$k_1 + 1$. Hence $\tau(n) = (k_1 + 1)$.

Theorem (4): Let $n = p_1^{k_1}\,p_2^{k_2}\,p_3^{k_3}\,\dots\dots\dots\dots p_r^{k_r}$, then

$$\tau(n) = (k_1 + 1)(k_2 + 1)(k_3 + 1)\dots\dots\dots\dots\dots(k_r + 1)$$

Proof: Since $gcd(p_i, p_j) = 1$, where $i \neq j$. Using theorem (2) and theorem (3) we get the required result i.e. $\tau(n) = (k_1 + 1)(k_2 + 1)(k_3 + 1)\dots\dots\dots\dots\dots(k_r + 1)$.

Corollary (2): Let $n = p^2$, then $\tau(n) = 3$.

Proof: Using theorem (3), we get required result. Similarly we can find other results by using theorem (3).

Result (1): Let $n = p^2\,q$, where $p \neq q$ then

$$\tau(n) = 6.$$

Result (2): Let $n = p^2\,q^2$, where $p \neq q$ then

$$\tau(n) = 9.$$

Result (3): Let $n = p^3\,q^1$, where $p \neq q$ then

$$\tau(n) = 8.$$

Result (4): Let $n = p^3 q^2$, where $p \neq q$ then

$$\tau(n) = 12 .$$

Result (5): Let $n = p^3 q^3$, where $p \neq q$ then

$$\tau(n) = 16 .$$

Result (6): Let $n = p^3$, where $p \neq q$ then

$$\tau(n) = 4 .$$

Result (7): Let $n = p^4$, where $p \neq q$ then

$$\tau(n) = 5 .$$

Result (8): Let $n = p^4 q^1$, where $p \neq q$ then

$$\tau(n) = 10 .$$

Result (9): Let $n = p^4 q^2$, where $p \neq q$ then

$$\tau(n) = 15.$$

Result (10): Let $n = p^5$, then

$$\tau(n) = 6 .$$

Result (11): Let $n = p^6$, then

$$\tau(n) = 7 .$$

Result (12): Let $n = p^5 q^1$, where $p \neq q$ then

$$\tau(n) = 12 .$$

Definition (2): Let $\tau(n)$ is the number of positive divisors of n, then $\tau^k(n)$ is also exist for all positive integers $k \geq 1$.

Verification of the above definition (2): For the verification of the above definition we construct a table (given below) for positive integers n, $1 \leq n \leq 100$.

n	$\tau(n)$	$\tau^2(n)$	$\tau^3(n)$	$\tau^4(n)$	$\tau^5(n)$	$\dots\dots\tau^k(n)$
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	2	2	2	2	2	2
4	3	2	2	2	2	2
5	2	2	2	2	2	2
6	4	3	2	2	2	2
7	2	2	2	2	2	2
8	4	3	2	2	2	2
9	3	2	2	2	2	2
10	4	3	2	2	2	2
11	2	2	2	2	2	2
12	6	4	3	2	2	2
13	2	2	2	2	2	2
14	4	3	2	2	2	2
15	4	3	2	2	2	2
16	5	2	2	2	2	2
17	2	2	2	2	2	2
18	6	4	3	2	2	2
19	2	2	2	2	2	2
20	6	4	3	2	2	2
21	4	3	2	2	2	2
22	4	3	2	2	2	2

23	2	2	2	2	2	2
24	8	4	3	2	2	2
25	3	2	2	2	2	2
26	4	3	2	2	2	2
27	4	3	2	2	2	2
28	6	4	3	2	2	2
29	2	2	2	2	2	2
n	τ(n)	$\tau^2(n)$	$\tau^3(n)$	$\tau^4(n)$	$\tau^5(n)$	$\dots\dots\tau^k(n)$
30	8	4	3	2	2	2
31	2	2	2	2	2	2
32	6	4	3	2	2	2
33	4	3	2	2	2	2
34	4	3	2	2	2	2
35	4	3	2	2	2	2
36	9	3	2	2	2	2
37	2	2	2	2	2	2
38	4	3	2	2	2	2
39	4	3	2	2	2	2
40	8	4	3	2	2	2
41	2	2	2	2	2	2
42	8	4	3	2	2	2
43	2	2	2	2	2	2
44	6	4	3	2	2	2

45	6	4	3	2	2	2
46	4	3	2	2	2	2
47	2	2	2	2	2	2
48	10	4	3	2	2	2
49	3	2	2	2	2	2
50	6	4	3	2	2	2
51	4	3	2	2	2	2
52	6	4	3	2	2	2
53	2	2	2	2	2	2
54	8	4	3	2	2	2
55	4	3	2	2	2	2
56	8	4	3	2	2	2
57	4	3	2	2	2	2
58	4	3	2	2	2	2
59	2	2	2	2	2	2
60	12	6	4	3	2	2
n	$\tau(n)$	$\tau^2(n)$	$\tau^3(n)$	$\tau^4(n)$	$\tau^5(n)$	$\dots\dots\tau^k(n)$
61	2	2	2	2	2	2
62	4	3	2	2	2	2
63	6	4	3	2	2	2
64	7	2	2	2	2	2
65	4	3	2	2	2	2
66	8	4	3	2	2	2

67	2	2	2	2	2	2
68	6	4	3	2	2	2
69	4	3	2	2	2	2
70	8	4	3	2	2	2
71	2	2	2	2	2	2
72	12	6	4	3	2	2
73	2	2	2	2	2	2
74	4	3	2	2	2	2
75	6	4	3	2	2	2
76	6	4	3	2	2	2
77	4	3	2	2	2	2
78	8	4	3	2	2	2
79	2	2	2	2	2	2
80	10	4	3	2	2	2
81	5	2	2	2	2	2
82	4	3	2	2	2	2
83	2	2	2	2	2	2
84	12	6	4	3	2	2
85	4	3	2	2	2	2
86	4	3	2	2	2	2
87	4	3	2	2	2	2
88	8	4	3	2	2	2
89	2	2	2	2	2	2

90	12	6	4	3	2	2
91	4	3	2	2	2	2
n	τ(n)	$\tau^2(n)$	$\tau^3(n)$	$\tau^4(n)$	$\tau^5(n)$	… … $\tau^k(n)$
92	6	4	3	2	2	2
93	4	3	2	2	2	2
94	4	3	2	2	2	2
95	4	3	2	2	2	2
96	12	6	4	3	2	2
97	2	2	2	2	2	2
98	6	4	3	2	2	2
99	6	4	3	2	2	2
100	9	3	2	2	2	2

From the above table we find the following results:

Result (13): If $n = 1$, then $\tau^k(n) = 1$, for all positive integers $k \geq 1$.

Result (14): If n is any prime number, then $\tau^k(n) = 2$, for all positive integers $k \geq 1$.

Result (15): If $n = p^2$, where p be any prime number. We find $\tau^k(n) = 2$, for all positive integers $k \geq 2$.

Result (16): If $n = pq$, where p *and* q are two prime numbers such that $p \neq q$. We find that $\tau^k(n) = 2$, for all positive integers $k \geq 3$.

Result (17): If $n = p^2 q$, where p *and* q are two prime numbers such that $p \neq q$. We find that $\tau^k(n) = 2$, for all positive integers $k \geq 4$.

Result (18): If $n = p^2\, q^2$, where $p\ and\ q$ are two prime numbers such that $p \neq q$. We find that $\tau^k(n) = 2$, for all positive integers $k \geq 3$.

Result (19): If $n = p^3\, q^1$, where $p\ and\ q$ are two prime numbers such that $p \neq q$. We find that $\tau^k(n) = 2$, for all positive integers $k \geq 4$.

Conclusions:

If n is any finite positive integer, we find out the following result $\tau^k(n) = 2$ for all positive integers $k \geq N$, where $1 \leq N \leq k$.

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