International Research Journal of Mathematics, Engineering and IT
Vol. 3, Issue 4, April 2016 IF- 3.563 ISSN: (2349-0322)

Website: www.aarf.asia Email : editor@aarf.asia , editoraarf@gmail.com

A NOTE ON Ω-OPEN SETS

C. K. GOEL[1] and POONAM AGARWAL[2]

[1]Department of Mathematics, Amity University, NOIDA.

[2]Department of Mathematics, Amity University, NOIDA.

ABSTRACT

*In this paper, for any given topological space (X,$\Im$), we introduce and study a new topology τ_Ω whose members we call **Ω**-open sets. We proved that $\Im_\Omega$ is not comparable to the given topology $\Im$. However, we investigate the behavior of **Ω**-open sets with respect to that of $\Im$- open sets in X.*

Keywords: **Ω**-open sets, **Ω**-closure, **Ω**-interior, Ω–O(X), Ω-continuity.

1. Introduction:

Let (X,$\Im$) be a topological space. In 1982, Hdeib [6] introduced the notion of ω-closeness. Using this concept, he introduced and studied ω-continuity. In 1966, the notions of θ-open subsets, θ-closed subsets and θ-closure were introduced by Veličko[15] for the purpose of studying an important class of topological spaces, namely, H-closed spaces in terms of filter bases. He also showed that the collection of θ-open sets in a topological space Xitself forms a topology $\tau_\theta$on X. Dickman and Porter [4], [5], Joseph [8]extended the work of Veličko to study further properties of H closed spaces. Noiri and Jafari[12], Caldas et al. [1] and [2], Steiner [13] and Cao et al [3] have also obtained several new and interesting results related to these sets. We use these concepts to define and develop a new class of open sets which we called **Ω-**open sets.

2. Preliminaries :

Let (X ,$\Im$) be a topological space. For a subset A$\subseteq$X, the closure and the interior of A is denoted by cl(A) and int(A), respectively. First let us recall some definitions, for any subset A of X, A is said to be

(i) ά-open[11] if A$\subset$int(cl(int(A))),

(ii) preopen[10] if A $\subset$int(cl(A)),

(iii) regular open [14] if A=int(cl(A)),

(iv) regular closed [14]) if A = cl(int(A))).

Definition 4.01 : A point x $\in$ X is said to be in the θ-closure [15] of a subset A of X, denoted by θ-cl(A), if cl(U) $\cap$ A $\neq$ φ for each open set U of X containing x.

Definition 4.02 : A point p is called a condensation point of A if every open set containing p contains uncountably many points of A. A subset A of a space X is called a ω-closed [6] if it contains all of its condensation points. The complement of ω-closed subset is called ω-open.

Notations 4.01 : The family of all ω-open (resp. θ-open, α-open) subsets of a space X is denoted by ω-O(X) (respectively, τ_θ= θ–O(X), α– O(X)).

Definition 4.03 : A subset A of a space X is called ω_θ-open[16] if for every x $\in$ A, there exists an open subset B $\subseteq$X containing x such that (B ~ θ-int(A)) is countable. As usual the complement of a ω_θ-open subset is called ω_θ-closed**.**

3. A New Class of Open Sets:

In this section, we introduce and study a new class of open sets.

Definition 5.01 : Let (X ,$\Im$) be a topological space and let A$\subseteq$X be any subset of X. A point p$\in$ X is called a **clocondensation** point of A if the closure of every open set containing p contains uncountably many points of A.

Example 5.01 : Let R be the set of real numbers endowed with the topology $\Im$={∅, R, Q´}where Q´ is the set of irrational numbers. Let A be an uncountable subset of Q´. Then each element of R is a clocondensation point of A. However, if B is any subset of Q, the set of rational numbers then B has no clocondensation point. In fact, if A is any uncountable subset of R, then each element of R is a clocondensation point of A.

Example 5.02 : Let R be the set of real numbers equipped with usual topology. Then each point of the interval (a,b) has only its own elements as clocondensation points. The set Q of rational numbers has no clocondensation point while the set Q´ of irrational numbers has each element of R as a clocondensation point.

Theorem 5.01 : Every uncountable subset of a second countable space (X ,$\mathfrak{I}$)has a clocondensation point.

Proof : Let $\mathfrak{B}$={U_n : $n \in$ N} be a countable base for the topology on X and let C be an uncountable subset of X. Let C not have any clocondensation point. Then for every x ∈ C there exists an open set V_x in X such that x ∈ V_x and cl(V_x)∩ E is countable. Now for every V_x there exists some U_{n_x}∈ such that x ∈ $U_{n_x} \subseteq V_x \subseteq$cl($V_x$)⟹ x ϵ U_{n_x}∩ E and U_{n_x}∩ E is countable. Now E = $\cup_{x \in C}$(U_{n_x} ∩ E). This shows that E is countable which yields a contradiction.

Theorem 5.02 : Every uncountable subset of a second countable space X has uncountably many clocondensation points.

Proof : Let A be the collection of all condensation points of an uncountable subset E of a second countable space X. If A is countable then E ~ A is uncountable and contains no clocondensation point of itself which contradicts Theorem 5.01.

Corollary 5.01 : Let (X ,$\mathfrak{I}$) be a second countable topological space and let A be an uncountable subset of X. Let B be the collection of all clocondensation points of A. Then E ~ A is countable.

Corollary 5.02 : Let(X ,$\mathfrak{I}$) be a second countable topological space and let A be a subset of X. If A does not contain any clocondensation points, then A is countable.

Theorem 5.03 : Every uncountable subset of a Lindelöf space (X ,$\mathfrak{I}$) has a clocondensation point.

Proof : Let (X ,$\mathfrak{I}$) be a Lindelöf space and let C be an uncountable subset of X such that C does not have any clocondensation point. Then for every x ∈X, there exists an open set V_x in X such that x∈ V_x and cl(V_x)∩ C is countable. Then {V_x : x∈ X} is an open cover of X. Since X is Lindelöf , it admits of a countable subcover ,say, {V_n : n ∈ N} then C ⊂(V_n∩ C) $\subset \cup_{n \in N}$ (cl(V_n) ∩ C) which implies that E is countable, a contradiction.

Corollary 5.03 : Every uncountable subset of a compact space (X ,$\mathfrak{I}$) has a clocondensation point.

Theorem 5.04 : The set of all clocondensation points of a subset of a topological space (X ,$\mathfrak{I}$) is closed.

Proof : Let X be a topological space and let A be a subset of X. Let D be the collection of all clocondensation points of A. Let $x \in X \sim D$. Then there exists an open set U in X such that $x \in U$ and $cl(U) \cap A$ is countable. We claim that $U \cap D = \emptyset$. For if $y \in U \cap D$ then $y \in U$ and $y \in D$ which implies that y is a clocondensation point of A and therefore $cl(U) \cap A$ is uncountable, a contradiction.

Remark 5.01 : Let $(X, \Im)$ be a topological space and let A be a subset of X. Then every condensation point of A is a clocondensation point of A.

Definition 5.02 :A subset A of a topological space $(X, \Im)$ is said to be a **Ω**-closed [6] if it contains all its **clocondensation** points. The complement of a **Ω**-closed subset is called **Ω**-open.

Example 5.03 : Let R be the set of real numbers equipped with the topology $\tau = \{\emptyset, R, Q'\}$ where Q' is the set of irrational numbers. Then the set $A = Q' \cup \{0\}$ is **Ω** –open since $(Q' \cup \{0\})' = Q \sim \{0\}$ is **Ω** closed, for if $x \in Q' \cup \{0\}$ then the closure of any neighborhood of x cannot intersect $Q \sim \{0\}$ in uncountably many points as $Q \sim \{0\}$ is itself countable. In general any set of the type $Q' \cup A$ where $A \subset Q$ is **Ω** –open in X, since any subset of Q is **Ω** –closed in X.

Theorem 5.05 : Let X be a topological space. Let $A \subseteq X$ then, A is **Ω** – open if for every $x \in A$ there exists an open set U such that $(cl(U) \sim A)$ is countable.

Proof : Let A be **Ω**-open then $X \sim A$ is **Ω**- closed. Let $x \in A$ then $x \notin (X \sim A)$. Since $X \sim A$ is **Ω** –closed, there exists an open subset U of X such that $x \in U$ and $cl(U) \cap (X \sim A)$ is countable which implies that $(cl(U) \sim A)$ is countable.

Conversely, let for every $x \in A$ there exist an open set U in X such that $x \in U$ and $(cl(U) \sim A)$ is countable. Now if $x \notin (X \sim A)$, then $x \in A$ and so there exists an open set U of X such that $x \in U$ and $(cl(U) \sim A)$ is countable, i.e., $cl(U) \cap (X \sim A)$ is countable. This shows that $x \notin$ **Ω**–$cl(X \sim A)$ implying that $(X \sim A)$ is **Ω**-closed and hence A is **Ω**-open.

Theorem 5.06 : Let $(X, \Im)$ be a topological space. Let Ω–$O(X)$ be the collection of Ω –open sets. Then Ω–$O(X)$ forms a topology on X. We denote this topology by $\Im_\Omega$ and call the resulting space as $\Im_\Omega$ -topological space.

Notation 5.01 : Let $(X, \Im_\Omega)$ be the topological space corresponding to a topological space $(X, \Im)$ and let $A \subseteq X$ be any subset of X. Then

(i) $\Im_\Omega$-closure of A is denoted by **Ω-** cl(A) .

(ii) $\Im_\Omega$- interior of A is denoted by **Ω-** int(A).

Definition 5.03 : The **Ω-**closure of A is the collection of all clocondensation points of A alongwith the elements of A i.e. **Ω-** cl(A) = A∪ {collection of all clocondensation points of A}.

Definition 5.04 : The **Ω-**interior of A is the collection of all those points x of A for which there exists an open neighborhood U_x such that $(cl(U_x) - A)$ is countable.

Theorem 5.07 : Let (X ,ℑ) be a topological space. Let Ω–C(X) be the collection of Ω –closed sets. Then it is obvious that Ω–C(X) is closed under arbitrary intersections and finite unions. Also ∅ and X are in Ω ~ C(X).

Example 5.04 : Let (R,𝔘) be the set of real numbers with usual topology. Let N be the set of natural numbers equipped with the relativized topology $\mathfrak{U}_N$. Obviously, $(N,\mathfrak{U}_N)$ is a discrete topological space. Let A⊆ N. Then, for every x∈N and for all U∈ℑ such that x∈ U, we have cl(U) ∩A is countable . Since A itself is countable hence x ∉ **Ω-** cl(A) implying that A is Ω-closed. Hence each subset of N is Ω-closed & therefore Ω–open which shows that the space $(N, \mathfrak{U}_N)$ is also discrete.

Example 5.05 : Let (R, 𝔘) be the set of real numbers with usual topology 𝔘. Let (a,b) ∈ 𝔘. Let x∈(a,b) then, if U=(a,b) we have , x∈ U, U ∈𝔘 and cl(U) ~ (a,b)={a,b} which is countable. Hence for this space every open set is Ω-open.

Example 5.06 : Let (Q,ℑ) be the set of rational numbers endowed with the indiscrete topologyℑ. Let A⊂ Q. Let x∉ A. Then x has the only neighbourhood Q with cl(Q)=Q and cl(Q) ∩ A is countable which shows that A is Ω–closed and hence each subset of Q is Ω–closed . Consequently A is Ω–open . This shows that $(Q ,\Im_\Omega))$ is a discrete topology.

Definition 5.05 : The intersection of all Ω-closed sets of X containing a subset A⊂ X is defined as the $\Im_\Omega$-closure of A.

Definition 5.06 : The union of all Ω-open sets of X contained in A⊂ X is defined as the$\Im_\Omega$-interior of A.

Lemma 5.02 : Let A be a subset of a space (X ,ℑ). Then the following hold:

(i) A is $\Im_\Omega$–closed in X if and only if A= $\Im_\Omega$–cl(A).

(ii) $\Im_\Omega$–cl (X ~ A) = X ~$\Im_\Omega$–int (A)

(iii) $\Im_\Omega$–cl(A) is $\Im_\Omega$–closed in X.

(iv) X∈$\Im_\Omega$-cl(A) if and only if A ∩ G ≠∅ ∀$\Im_\Omega$-open sets G containing A.

Theorem 5.07: If a $\Im$–open set U has countably many boundary elements then U is $\Im_\Omega$-open also.

Proof : Let U be a $\Im$– open set with countably many boundary points. Then U will be $\Im_\Omega$ –open if for every x∈U there exists V∈$\Im$ such that x∈V and cl(V)~ V = countable. Take V=U then, cl(U)~U is countable. Hence U is $\Im_\Omega$-open.

Theorem 5.08 : If A is both $\Im$–open and $\Im$–closed, then A is both $\Im_\Omega$–open and $\Im_\Omega$–closed.

Proof : A is $\Im$–closed ⟹ A =cl(A). Now A is $\Im_\Omega$- open if for every x∈A there exists U∈$\Im$ such that x∈U and cl(U)~ A is countable. Let U=A (because A is $\Im$–open also) ⟹ cl(A)~ A =∅ which is obviously countable. Thus A is $\Im_\Omega$–open. Now A is $\Im$–closed and$\Im$–open⟹ A is $\Im_\Omega$–open because A is open and closed ⟹ X ~ A is closed and open⟹ X ~ A is also $\Im_\Omega$–open⟹ A is $\Im_\Omega$–closed also.

Theorem 5.09 : Every countable set is $\Im_\Omega$-closed.

Proof : Let A be a countable set. Let x∉A and let U be an open set containing x then, A∩ clU is countable which implies that x ∉ $\Im_\Omega$ – cl(A)⟹ A is $\Im_\Omega$-closed.

Theorem 5.10 : A subset A is an $\Im_\Omega$–open set for a space (X , $\Im$) if and only if there exists a $\Im$–open set U and a countable set V such that for every x ∈ A , we have x ∈ U and (cl(U)~ V) ⊂A.

Proof : Let x∈A ⟹there exists U∈$\Im$ such that x∈ U and cl(U) ~ A is countable. If we take cl(U)~A=V then cl(U)~V⊂A.Conversely, let x ∈ A. Then there exists a $\Im$–open subset U containing x and a countable subset V such that cl(U)~V⊂A. But this shows that cl(U) ~ A is countable implying the result.

Definition 5.07 : A space (X,$\Im$) is said to be anti locally countable space if nonempty open sets are uncountable.

Theorem 5.11 : If (X,$\Im$) is a anti locally countable space then so is (X, $\Im_\Omega$).

Proof : Let A∈ $\Im_\Omega$ and let x∈ A. Then, there exists a $\Im$– open subset U⊂X and a countable set V containing x satisfying cl(U)~ V⊂A. Now U∈$\Im$ implies that U is uncountable and so cl(U) ~ V is uncountable. Hence A is uncountable.

Theorem 5.12 : If X is an antilocally countable regular space and if A is a $\Im$–open, then, cl(A) ⊂ $\Im_\Omega$- cl(A).

Proof : Let $x \in cl(A)$. We show that $x \in \Im_\Omega$ - $cl(A)$.Let B be an $\Im_\Omega$–open set containing x. Then, there exists $U \in \Im$ and a countable set V with $x \in U$ and is such that $(cl(U) \sim V) \subset B \Rightarrow (cl(U) \sim V) \cap A \subset B \cap A. \Rightarrow (cl(U) \cap A) \sim V \subset B \cap A$. Now, $x \in U$ and U is an open set, therefore $x \in cl(A) \Rightarrow U \cap A \neq \emptyset$. Further, since both U and $A \in \Im$, we have that $U \cap A$ is Uncountable. But this implies that $cl(U) \cap A$ is uncountable $\Rightarrow B \cap A$ is uncountable $\Rightarrow B \cap A \neq \emptyset \Rightarrow x \in \Im_\Omega$ - $cl(A)$

Theorem 5.13 : If $(X, \Im)$ is a regular space then, $\Im_\Omega$ - $cl(A) \subset cl(A)$.

Proof : Suppose $x \in \Im_\Omega$ - $cl(A)$, then for every $U \in \Im$ with $x \in U$ we have that $cl(U) \cap A$ is uncountable. Now let V be an open set containing x. Since X is regular, there exists an open set V_1 such that $x \in V_1 \subset cl(V_1) \subset U$. Now because $x \in \Im_\Omega$ - $cl(A)$ we have that $cl(V_1) \cap A$ is uncountable $\Rightarrow U \cap A$ is uncountable. Hence, $U \cap A \neq \emptyset \Rightarrow x \in cl(A). \Rightarrow \Im_\Omega$ - $cl(A) \subset cl(A)$.

Corollary5.04: If $(X, \Im)$ is anti locally countable regular space then $\Im_\Omega$ - $cl(A) = cl(A)$.

4. Relation of Ω open sets with some other kind of open sets

RESULT 6.01 : The topology $\tau\Im_\omega$ is finer than the topology $\tau\Im_\Omega$**.**

RESULT 6.02 : The topology $\Im_\Omega$ is finer than the topology $\Im_\theta$**.**

RESULT 6.03 : In a regular topological space, the topology $\Im_\Omega$ is finer than the topology $\Im$**.**

RESULT 6.04 : In a regular topological space, the topology τ_Ω is finer than the topology $\Im_\omega$**.**

5. The Ω-continuity

DEFINITION 7.01 : A function $f : X \to Y$ is said to be $\Im_\Omega$-continuous if $\forall x \in X$ and $\forall$ open sets V in Y containing f(x), $\exists$ an $\Im_\Omega$-open subset U in X such that $x \in U$ and $f(U) \subset V$.

THEOREM 7.01 : For a function $f : X \to Y$, the following are equivalent :

(i) f is $\Im_\Omega$-continuous.

(ii) $f^{-1}(A)$ is $\Im_\Omega$-open in X $\forall$ open subsets A in Y.

(iii) $f^{-1}(K)$ is $\Im_\Omega$-closed in X $\forall$closed subsets A in Y.

THEOREM 7.02 : Following are equivalent for a function $f : (X,\Im) \to (Y,\sigma)$

(i) f is Ω - continuous.

(ii) $f: (X, \Im_\Omega) \to (Y, \sigma)$ is continuous.

EXAMPLE 7.01 : Let R be the real line equipped with the topology $\Im = \{ \emptyset, R, Q' \}$ where Q´ is the set of irrational numbers. Let Y = {a,b,c,d} and let σ = (Y, ∅,{c}, {d},{a,c}, {c,d},{a,c,d}} be a topology on Y. Define a function $f: (R,\Im) \to (Y,\sigma)$ as :

$$f(x) = \begin{cases} a & \text{if } x \in Q' \cup \{0\} \\ b & \text{if } x \notin Q' \cup \{0\} \end{cases}$$

then it can easily be seen that f is $\Im_{\Omega}$ – continuous.

EXAMPLE 7.02 : Consider $(N, \Im_D)$ where $\Im_D$ denotes the discrete topology on set N of natural numbers then $\Omega O(N) = \Im_D$ and $f : (N. \Im_{\Omega}) \to (N. \tau_0)$ defined by $f(x) = x \ \forall x \in N$ is continuous.

Acknowledgement:

We are grateful to Prof. M. K. Gupta, C.C.S. University Meerut for providing his valuable guidance and necessary facilities in completing the work.

References:

Journal Papers

[1] Caldas, M., Jafari, S. and Kovar, M.M, Some properties of θ-open sets, Divulg. Mat., 12 (2) (2004), 161-169.

[2] Caldas, M., Georgiou, D.N., Jafari, S. and Noiri, T., On (Λ; θ)- closed sets, Q and A in General Topology, 23 (2005), 69-87.

[3] Cao, J., Ganster, M., Reilly, I. and Steiner, M., δ-closure, θ-closure and generalized closed sets, Applied General Topology, 6 (1) (2005), 79-86.

[4] Dickman Jr., R.F. and Porter, J.R., θ-closed subsets of Hausdorff spaces, Pacific J. Math., 59 (1975), 407-415.

[5] Dickman Jr., R.F. and Porter, J.R., θ-perfect and θ-absolutely closed functions, Ilinois J. Math., 21 (1977), 42-60.

[6] Hdeib, H.Z., ω-closed mappings, Rev. Colombiana Mat., 16 (1982), no. 1-2, 65-78.

[7] Isbell, J.R., Uniform spaces, Amer. Math. Soc. Math. Surveys, no. 12 (1964).

[8] Joseph, J.E., θ-closure and θ-subclosed graphs, Math. Chronicle, 8 (1979), 99-117.

[9] Long, P.E. and Herrington, L.L., The τ_θ-topology and faintly continuous functions, Kyungpook Math. J., 22 (1982), 7-14.

[10] Mashhour, A.S., AbdEl-Monsef, M.E. and El-Deeb, S.N., On pre-continuous and weak pre-continuousmapppings, Proc. Math. Phys., Soc. Egypt, 53 (1982), 47-53.

[11] Njastad, O., On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961-970.

[12] Noiri, T. and Jafari, S., Properties of (θ; s)-continuous functions, Topology and its Applications, 123 (2002), 167-179.

[13] Steiner, M., Verallgemeinerteabgeschlossene Mengen in topologischen Räume, Master Thesis, Graz University of Technology, Austria, 2003.

[14] Stone, M.H., Applications of the theory of Boolean rings to general topology, Trans. Amer Math Soc., 41 (1937), 375-381.

[15] Veličko, N.V., H-closed topological spaces, Mat. Sb., 70 (1966), 98-112; English transl. (2), in Amer. Math. Soc. Transl., 78 (1968), 102-118.

[16] Ekici E.,Jafari S., Latif R.M., On a finer topological space than τ_θ, Italian Journal of Pure And Applied Mathematics- N. 27(2010),293-304.