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STRUCTURE OF G-TYPE SPACES

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Abstract. *The issue of G-structures founded by Goldman in 1951 and then its extension named"G-type structures"was raised in 2012 by Karamzadeh and Moslemi. In this paper is expressed the applications of G-type structures in spectral spaces, which in for G-type domain R has been introduced a new domain saying"pullback of G-type domain"with title of* $\tilde{R}$. *It has been proven, if R is a G-type domain then* $Spec(\tilde{R})$ *homomorphic to Spec(R) and in special if R is a saturated G-type domain and* $S^{-1}R \subset R^*$ *.then Ris coincides to* $\tilde{R}$.

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1. Introduction

The properties of Hilbert ring and Hilbert Nullstellensatz was the one of important concepts raised by Goldman in 1951, this purpose were defined as a new structure by the title of*"G-structures"*,the main idea was been the applications of these structures in Hilbert rings, these concepts as a suitable classified form have come in commutative algebra of Kaplansky[13].
After a long time was expressed a new concept of extension of these structures with the title of*"G-type structures"*by Karamzadeh and Moslemi in 2012[14] , where in was pointed the suitable and broader of Hilbert Nullstellensatz, on this way G-structures, G-type domains and G-type ideals were defined and also some Theorems and Corollaries were presented.
In this paper is discovered the applying of G-type structures to spectral spaces by instruction of the historical concepts.

So firstly, the G-type domains and G-type ideals are defined, then by paper [14] some important Theorems are come and finally after the presenting a few Lemmas is proved some important Theorems as the following:
A Noetherian domain *R* is a G-type domain if and only if it has just countable number of nonzero minimal prime ideals. In addition, if *R* is a G-type domain then $Spec(\tilde{R})$ homomorphic to *Spec(R)*.

2. Mathematical Notations

Definition 2.1 A commutative ring with unit in which every finitely generated ideal is principal is called a Bézout ring, if a Bézoutring has no zero divisors it is called a Bézout domain.
If each finitely generated ideal of an integral domain R isinvertible, then it's called a prüfer domain.

Lemma 2.2.[14] Let R be a domain with quotient field K, R is said to be aG-domain if K is a finite type R-algebra.

Lemma 2.3. *[15]Let* $P(R) = \bigcap_{P \in Spec\,(R)_{P \neq 0}} P$ *,(pseudo-radicalof R), R is a G-domain if and only if* $P(R) \neq 0$.
In addition Spec(R) is finite set then R is evidently a G-domain.

Definition 2.4. A domain R is called a G-type Domain if its quotient field is countably generated as a R-algebra.
R is a G-type Domain if and only if its zero ideal is thecontraction of a maximal ideal in $R[x_1 .\ x_2\ .\ \cdots . x_n . \cdots]$.
A prime ideal I of $R[x_1 .\ x_2\ .\ \cdots . x_n . \cdots]$ is G-type if and onlyif its contraction in R and $R[x_1 .\ x_2\ .\ \cdots . x_n]$for all $n \geq 1$are G-type.

Theorem 2.5.[14]*Let P be a prime ideal in a ring R, then the following areequivalent:*
i) P is a G-type ideal in R.
ii) There is a countable multiplicative closed set $S \subseteq R$ *such that: P is maximal with respect to having the empty intersection with S.*
iii) There are either only a countable number of prime ideals in R/P *orany uncountable set of prime ideals properly containing P, say F,can be written in the form* $F = \cup_{n \in \Lambda} F_n$ *, where* Λ*is a subset of the natural numbers, P is properly contained in* $\cup_{Q \in F_n} Q$*for each n and some of the* F_n *are uncountable.*

[2] **Corollary 2.6.** *Let R be a domain, such that each of its ideals countably generated ,then R is a G-type domain if and only if there exists a countably generated R-algebra"T"contains the quotient field of R.*

Theorem 2.7. [16] *If R be a countable domain, then there is a maximal ideal M in* $R[x_1 . x_2 . \cdots . x_n . \cdots]$*such that* $M \cup R = (0)$ *and each* $x_n + M$ *is algebraic over* $\frac{R+M}{M} \cong R$.

Corollary 2.8. [2] *Let R is a domain, R is a G-type domain if and only if there exists amaximal ideal M in* $R[x_1 . x_2 . \cdots . x_n . \cdots]$ *such that* $M \cup R = (0)$.

Corollary 2.9. [2] *Let K be an algebraically closed field and* $R = K[x_1 . x_2 . \cdots . x_n . \cdots]$*then each maximal ideal M of R is of the form* $M = (x_1 - \alpha_1 . x_2 - \alpha_2 . \cdots)$ *if and only if K is uncountable.*

Definition 2.10 Let *R* be a ring, then:
i) *dim R* = the supremum of all lengths of chain of distinctprime ideals in *R.*
ii) Let *M* be an R-module, the Krull dimension of M , which isdenoted by "*k-dim M*", is defined by transfinite recursion asfollows: *k-dim M = -1* if *M =(0)*and for every ordinal

number of α , we say that $k-dim\,M = \alpha$ if $k-dim \not< \alpha$ andgiven any infinite descending chain "$M_1 \supseteq M_2 \supseteq \cdots$" of submodules in M there exists some k such that: $k-dim\,M_m/M_{m+1} < \alpha$ for all $m \geq k$.
The Krulldimension of a ring *R*," *k-dimR*",is defined to be the Krull dimension of a right *R*-module *R*.

Theorem 2.11. [14]*Let R be a Noetherian domain, R is a G-domain if and only if R issemilocal and* $k-dim\,R \leq 1$.

Remark 2.12. [3] The ring of *R* is said has the*"CPA"*property(*Countable PrimeAvoidance*) if $A \subseteq \bigcup_{i=1}^{\infty} p_i$ (A an idealof R) then $A \subseteq P_i$. $\exists i$.
Theorem 2.13. [2] *Let R be a complete Noetherian semi-local ring, then a primeideal P of R is a G-type ideal if and only if R is a G-type idealif and only if it is a G-ideal.*

Theorem 2.14. [14] *Let R has countable Noetherian dimension, then R is a finitedirect sum of G-type domain if and only if each localization* R_P *is a G-type domain or countably generated as a* $\phi_P(R)-algebra$ *,where* $\phi_P: R \longrightarrow R_P$ *is the natural homomorphism.*

Definition 2.15. Let *R* be a ring and *X* is the set of all prime ideals of *R*, let $E \subseteq R$, if we define *V(E)* as follows:

$$V(E) = \{P \in X : P \supseteq E\}$$

Then :
i)V(0)=X ,$V(1) = \emptyset$.
ii) If $(E_i)_{i \in I}$ be every family of subsets of *R*, then:

$$V(\cup_{i \in I} E_i) = \cap_{i \in I} V(E_i)$$

iii)$V(a \cap b) = V(ab) = V(a) \cup V(b)$,*a* and *b* are arbitrary ideals of *R*.

Note 2.16.*1)* The set of *V(E)* is satisfying all the axioms of closed sets in atopological space, which is called the Zariski topology.
2) A topological space *X* is called, prime spectrum of *R* and it'swritten by *Spec(R).*

Definition 2.17. Let $\forall f \in R$. X_f be the complement of *V(f)* in the *X=spec(R)*, so the sets X_f are open, therefore they form abasis of open sets for the Zariski topology, which are:
1) $X_f \cap X_g = X_{fg}$
2) If $X_f = \emptyset$ then *f* is nil potent.
3) $X_f = X \Leftrightarrow f$ is a unit.\\
4) $X_f = X_g \Leftrightarrow V(<f>) = V(<g>)$
5) *X* is quasi-compact (that is every open covering of *X* has afinite sub covering).
6) Furthermore, each $X_f$is also quasi-compact.
7) An open subset of *X* is quasi-compact if and only if it is afinite union of sets X_f.

Note 2.18. [3] The sets X_f are called basic open sets of $X = Spec(R)$. A topological space *X* is said to be irreducible either $X \neq \emptyset$ or every pair of non-empty open sets in *X* intersect. Equivalently if every non-empty open set is dense in *X*, therefore *Spec(R)*is irreducible if and only if the nil radical of *R* isa prime ideal.

Remark 2.19. [6] If *R* be a ring and *X=Spec(R)* , then the irreducible componentsof *X* are the closed sets *V(P),*where P is a minimal prime ideal of*R*.Let $R = \prod_{i=1}^{n} R_i$ be the direct product

of rings R_i , so *Spec(R)* is the disjoint union of open (and closed) subspacesX_i,where X_i is canonically homeomorphic with $Spec(R_i)$.
Conversely, Let R be any ring ,the following are equivalent:
i) *X= Spec(R)* is disconnected.
ii) $R \simeq R_1 \times R_2$, where any of the rings $R_1 . R_2$aren't the zero ring.
iii) *R* contains an idempotent not equal to 0 , 1.

Note 2.20. Let *R* is a Boolean ring $X = Spec(R)$,then:
i) For each $f \in R$, the set X_f is both open and closed in*X*.
ii) Let $\{f_1 . f_2 . \cdots . f_n\} \in R$, then $X_{f_1} \cup \cdots \cup X_{f_n} = X_f$. $\exists f \in R$.
iii) The sets X_f are the only subsets of *X* those are both openand closed.
iv) X is a compact Hausdorff space.

Definition 2.21.Let *R* be a domain with quotient field *K* and *P* be any prime idealof *R* and $S = R - P$ be a *"mcs"(Multiplicative Closed Subset)* of *R* and $\bar{R}$bethe integralclosure of *R* and *T* be the ring of fraction of *R* so:
i) $\tilde{R}$ is a pullback of a ring of fraction *T* of *R* suchthat each nonzero prime of *T* is contained in the union of height *1*primes.
ii)R^+ : the seminormalization of *R*.
iii)R' : the integral closure of *R*.
iv)R^* : the complete integral closure of *R*.
v) Let $P(R) = \bigcap_{P \in Spec\,(R)_{P \neq (0)}} P$, it's shown thatfor brevity by *P*, so it's defined as following:
1) P^+: $seminormalization\ of\ P$.
2) P': $integral\ closure\ of\ P$.
3) P^*: $complete\ integral\ closure\ of\ P$.
vi)X(R): denote the set of all valuation overrings of *R*.
$X^1(R)$: The set of all one-dimensional valuation overringsof *R*.
vii)m_V:denote the maximal ideal of any given valuation ring*V*.

Theorem 2.22. [6] *Let R be a G-domain. Then, the following are equivalent:*
1) $\{ x \in K | x^2 \in P . x^3 \in P\} \subset R$.
2) $P = P^+$.
3) $P = P'$.
4) $P = P^*$.
5) $\cap \{m_V | V \in X^1(R)\} \subset R$.

The G-domain R is called saturated if it satisfies in each of equivalentconditions of Theorem"2.22".

Corollary 2.23. [7] *i) If R is a seminormal G-domain, then R is saturated if and only if*$P(R) = P(R') = P(R^*)$.
ii) *For a saturated G-domain R,* $P(R) = P(S^{-1}R)$ *if and only if*$S^{-1}R \subset R^*$.
iii) *If R is a saturated G-domain, then* $R^* = \cap \{ V | V \in X^1(R)\}$ *and is completely integrally closed.*

Lemma 2.24. [6] A pullback diagram of commutative rings

$$\begin{array}{ccc} A \times_T R & \overset{\pi_1}{\rightarrow} & A \\ \pi_2 \downarrow & & \phi_2 \downarrow \\ R & \overset{\phi_1}{\rightarrow} & T \end{array}$$

where ϕ_1 is surjective, naturally gives rise to acommutative diagram

$$\begin{array}{ccc} Spec(A \times_T R) & \longleftarrow \mu_1 \longleftarrow & Spec(A) \\ \mu_2 \uparrow & & Spec(\phi_2) \uparrow \\ Spec(R) & \longleftarrow Spec(\phi_1) \longleftarrow & Spec(T) \end{array}$$

in such a way that $Spec(A \times_T R)$ is identified with thetopological space $Spec(A) \cup_{Spec\,(T)} Spec(R)$ via the mapsμ_1 *and* μ_2. Moreover, π_1 is a surjective map andμ_1 gives a closed embedding of $Spec(A)$ into$Spec(A \times_T R)$.

3. Some Important Properties of G-type Domains

Definition 3.1. The *G*-type domain *R* is called essential (i.e.,a *G*-type domain ofessential type) if each nonzero prime ideal of *R* is contained inthe union of the height 1 prime ideals of *R*. Each *R* which is *G*-type domain with one dimensional such that itsmotivation $S^{-1}R$ is essential, so that $\tilde{R}$ is apullback of an essential *G*-type domain.

Definition 3.2. For each commutative ring R, let $Spec^i(A)$ denote the subspaceof $Spec(R)$ consisting of the height *i* primes.In particular if $Spec^1(R) = \{P_1 . P_2 . \cdots . P_n\}$, then: $S^{-1}R = \cap R_{p_i}$.

Lemma 3.3.*Let R be a G-type domain, then:*
i) Every overring of R is a G-type domain, in particular $S^{-1}R$*is a G-type domain.*
ii)$P(S^{-1}R) = S^{-1}(P(R))$
iii) $S^{-1}(R) \subset \cap \{R_Q | Q \in Spec^1(R)\}$
Proof. i) It is concluded immediately from definition of a G-typedomain.
ii) Since we have $P(R) = \cap \{Q | Q \in Spec^1(R)\}$ and also

$$P(S^{-1}R) = \cap \{S^{-1}Q | S^{-1}Q \in Spec^1(S^{-1}R)\}$$
$$= \cap \{S^{-1}Q | Q \in Spec^1(R) . Q \cap S = \emptyset\}$$

But $S = R - \cup \{Q | Q \in Spec^1(R)\}$ implies that $Q \cap S = \emptyset$ is true for every $Q \in Spec^1(R)$ .thus:

$$P(S^{-1}R) = \cap \{S^{-1}Q | Q \in Spec^1(R)\}$$

$$\supset S^{-1}(\cap \{Q | Q \in Spec^1(R)\}) = S^{-1}(P(R)).$$

To verify thereverse containment, let:

$$x = \frac{x_1}{x_2} \in P(S^{-1}R) \subset S^{-1}(R) . \ \ where \ \ x_1 \in R \ \ and \ \ x_2 \in S$$

Since x $\in \cap \{S^{-1}Q | Q \in Spec^1(R)\}$, it follows that, for every $Q \in Spec^1(R)$ there exists$s_Q \in S$ suchthat $s_Q x \in Q$. Then since $s_Q x_1 = s_Q x_2 x \in Q$, so we have$x_1 \in Q$ for every $Q \in Spec^1(R)$ andtherefore,$x \in S^{-1}(\cap \{Q | Q \in Spec^1(R)\})$ as claimed.
iii) Since $S \subset R/Q$ for every $Q \in Spec^1(R)$ thereforethe proof is completed. □

Lemma 3.4.*Let R be a G-type domain, then:*

i) Every valuation overring of R other than K is contained in amaximal valuation overring of R distinct from K.
ii) Every $0 \neq Q \in Spec(R)$ *contains a minimal nonzeroprime, therefore:*

$$P(R) = \cap \{P|P \in Spec^1(R)\}$$

iii) $$P(R) = \cap \{ (m_V \cap R)|V \in X^1(R)\}$$

Proof.i) Since each overring of an *G*-type domain is alsoan *G*-type domain and each union of *G*-type domains is an *G*-typedomain, therefore by Zorn's lemma, let $\{R_\alpha\}$ be achain of valuation rings in $X(R) - \{K\}$, then $W = \cup R_\alpha$ is necessarily a valuation overring of *R*.Now let $0 \neq x \in P(R)$,since x lies in every nonzero prime ideal of*R*, then $1/x \notin R$ for every nontrivial $R \neq X(R)$therefore$1/x \notin W$ and hence $W \neq K$.
ii) By Zorn's lemma, if $\{ P_\alpha\}$ be a chain ofprime ideals in $Spec(R) - \{0\}$,then $Q = \cap P_\alpha$ alsois a prime ideal, since $0 \neq P(R) \subset P_\alpha$. $\forall \alpha$ .Therefore $P(R) \subset Q$ and hence $Q \neq 0$.
iii) Let $P \in Spec^1(R)$be an arbitrary prime ideal, thereexists $V \in X(R)$ such that $V \subset W$. Thus $m_V \cap R$ is anonzero prime inside *P*, whence $m_W \cap R = P$. □

Corollary 3.5.*For every G-type domain R we have:*

$$\frac{S^{-1}R}{P(S^{-1}R)} \simeq S^{-1}(\bar{R})$$

Definition 3.6. Let *R* be any ring. We denote by *ZD(R)* (respectively, *NZD(R)* the setof all zero divisors (all nonzero divisors) of *R*.the totalquotient ring of *R*,denoted *Tot(R)*,is: $\{r/s|r \in R \quad and \quad s \in NZD(R)\}$.

Lemma 3.7. *Let R be an G-type domain with pseudo-radical P, and let Y be theunion of all minimal primes of* $\bar{R} = R/P$*.Then:*
i) $Y = ZD(\bar{R})$
ii) $Tot(\bar{R}) = S^{-1}(\bar{R}) \simeq (S^{-1}R)/(P(S^{-1}R))$.

Proof.i) That $Y \subset ZD(\bar{R})$ is well known. For the reverse, note that:
$P = \cap \{Q|Q \in Spec^1(R)\}$.
Thus, if $\bar{x}\bar{y} = 0$ and$\bar{y} \neq 0$, then there exists $Q \in Spec^1(R)$ such that$y \notin Q$. It follows that $x \in Q$ and so$\bar{x} \in Y$.
ii) Let $\bar{x} \in \bar{R}$. By (i) $\bar{x} \in NZD(R)$ if andonly if $x \neq \cup \{Q|Q \in Spec^1(R)\}$ andif and only if $x \in S$. Thus $Tot(\bar{R}) = S^{-1}(\bar{R})$. Therefore:
$S^{-1}(\bar{R}) \simeq (S^{-1}R)/(P(S^{-1}R))$ □

Note 3.8. By the proof of last lemma a *G*-type domain R has essential type ifand only if $\bar{R} = Tot(\bar{R})$.Thus every one dimensional*G*-type domain has essential type. Since it is Known that all Noetherian and all Krull *G*-type domain satisfy $dim(R) \leq 1$,it follows that all Noetherian and Krull *G*-type domains haveessential type.In addition each valuation ring *V* of finite dimension $n \geq 2$ is a G-type domain of nonessential type (indeed, the pseudo-radical of *V* is the unique height 1 prime *P* of *V* and so $S^{-1}V = V_P \neq V$(.

Theorem 3.9. *Let R be an integrally closed G-type domain. then:*
i) $R^* = \cap \{V|V \in X^1(R)\}$
ii) $P(R) = P(R^*) = \cap \{m_V|V \in X^1(R)\}$

Proof. i) This is a similar result due to Gillmer and Heinzer[11].

ii) By lemma "3.4" (iii) $P(R^*) = \cap \{(m_V \cap R^*) | V \in X^1(R^*)\}$ and by the part of (i) $X^1(R) = X^1(R^*)$, therefore:

$$P(R^*) = (\cap \{m_V | V \in X^1(R)\}) \cap R^* .$$

On the other hand, applying Lemma "3.4"(iii) to R yields:
$P(R) = \cap \{(m_V \cap R) | V \in X^1(R)\} = (\cap \{m_V | V \in X^1(R)\}) \cap R.$
An application of Lemma "3.4"(i) makesclear that:

$$\cap \{m_V | V \in X^1(R)\} = \cap \{m_V | V \in X(R)\} \subset \cap \{V | V \in X(R)\} = R$$

(Since R is integrally closed).
The assertions now followeasily. □

Lemma 3.10. *Let a ring R have dcc on finite intersections of prime ideals (Rhas "dcc" (Descending Chain Conditions) on prime ideals, R/P has only a countable number ofnonzero minimal primes for each prime P), then each prime ideal Pof R is a G-ideal (G-type ideal).*

Proof. At first we must show that a domain with "*dcc*" onfinite intersections of prime ideals is G-domain. To see this,let *A* be minimal among the ideals which are finite intersections of nonzero prime ideals, (i.e., $(0) \neq A$ is in fact theintersection of all the nonzero prime ideals and we are through.For the other part, we must show that each domain *R* with dcc on prime ideals and having only a countable number of nonzero minimalprime ideals is *G*-type ideal. To see this , let $P_1, P_2, \ldots, P_n, \ldots$ be the nonzero minimal prime ideals of R andnote that each nonzero prime ideal contains one of $P_i^{'} s$. Nowfor each n, take $0 \neq a_n \in P_n$, and let *S* be the *"mcs"* setgenerated by $\{a_1, a_2, \ldots, a_n, \ldots\}$. It is now that $S \cap P \neq (0)$ for each nonzero prime ideal P and this completes theproof. □

Theorem 3.11. *Let R be a Noetherian domain, then R is a G-type domain if andonly if R has only a countable number of nonzero minimal primeideals*

Proof. If R has only a countable number of nonzerominimal prime ideals, then we are through by lemma of*"3.9"*.
Conversely, let $S = \{s_1, s_2, \ldots, s_n, \ldots\}$ be a countable *"mcs"*set such that $S \cap P \neq \emptyset$ for all nonzero primeideals *P*. Let us assume that the set of nonzero minimal primeideals is uncountable and drive a contradiction. Now there mustexist an element $s \in S$ such that *s* belongs to an uncountablenumber of nonzero minimal prime ideals. Clearly each of these prime ideals are minimal over *(s)* and it goes without saying that*(s)* is not a prime ideal. Now considering the Noetherian ring$R/(s)$ which has an infinite number of minimal prime ideals, itgives us the desired contradiction. □

Corollary 3.12.*Let $k - dimR = n$, then R is a G-type domain if and only if thenumber of nonzero minimal prime ideals in R are countable.*

Theorem 3.13. *Let R be a Noetherian domain with the CPA property, then R is aG-type domain if and only if Spec(R) is countable and each nonzeroprime ideal is maximal.*
(i.e., $k - dim\, R \leq 1$)

Proof. If Spec(R) is countable and $k - dim\, R \leq 1$, thenby Theorem*"3.11"R* is a G-type domain.
Conversely, we claim that every prime ideal has the rank less or equal one and by theorem of*"3.11"*, the proof is complete. So, let $P \in Spec(R)$ is a primeideal with $rank(P) \geq 2$ and derive a contradiction.

It is well-known that the rank of every prime ideal in Noetherian ringsis finite (i.e., we may assume that *rank (P)=n*). Hence there existsa chain of prime ideals.

$$P = P_n \supset P_{n-1} \supset \ldots \supset P_2 \supset P_1 \supset (0),\ of\ length "n",(i.e., rank(P_2) = 2)$$

In view of theorem of "9" there exist only countable numbers of primeideals of rank less than or equal to one. Thus we may assume that$P_1 = Q_1, Q_2, \ldots, Q_n, ..$ are the only primes between (0) andP_2. But by the *"CPA"* property we can have $P_2 \subseteq \cup_{i=1}^{\infty} Q_i$,so there exists $x \in P_2$ and then bythe "Principal Ideal Theorem" we have $rank(P_2) \leq 1$ whichis the desired contraction.□

Theorem 3.14.*i) If R is a G-type domain, then $Spec(\tilde{R})$ ishomeomorphic to Spec(R) (via the map induced by the naturalinclusion of Rin $\tilde{R}$).*
ii) If R is a saturated (e.g.,seminormal) G-type domain and $S^{-1}R \subset R^*$, then $\tilde{R} = R$ is a pullback type.

Proof. This Proof is similar to theorem of [7,thm.2.15]
i) By definition of G-type domain, since each overringof a G-type domain is also G-type domain and each pullback of aG-type domain is also an overring of it, so it is obviously a G-type domain. Now by pullback diagram of canonical homomorphism's:

$$\begin{array}{ccc} \tilde{R} & \overset{\pi_1}{\rightarrow} & \bar{R} \\ \pi_2 \downarrow & & \phi_2 \downarrow \\ S^{-1}R & \overset{\phi_1}{\rightarrow} & T \end{array}$$

We obtain a commutative diagram

$$\begin{array}{ccc} Spec(\tilde{R}) & \longleftarrow \mu_1 \longleftarrow & Spec(\bar{R}) \\ \mu_2 \uparrow & & \alpha_2 \uparrow \\ Spec(S^{-1}R) & \longleftarrow \alpha_1 \longleftarrow & Spec(T) \end{array}$$

which is $Spec(\tilde{R})$ is identified with $Spec(\bar{R}) \cup_{Spec\,(T)} Spec(S^{-1}R)$ and μ_1 is a closedembedding. The map α_1,being induced by the surjectionϕ_1,is just the standard correspondence between prime idealsin $T = (S^{-1}R)/(P(S^{-1}R))$ and prime ideals in $S^{-1}R$ thatcontain $P(S^{-1}R)$. Since each nonzero prime contains thepseudo-radical, the image of α_1 is $Spec(S^{-1}R)\backslash\{0\}$. But every $\alpha_1(P)$ in thisimage is identified with the corresponding $\alpha_2(P)$ in$Spec(\bar{R})$.
Thus, up to homeomorphism,$Spec(\bar{R}) \cup_{Spec\,(T)} Spec(S^{-1}R) = Spec(\bar{R} \cup \{0\}$
(the second union being disjoint).
Moreover, since $\mu_1$is a closed embedding, Spec($\bar{R}$) is a closed set in $Spec(\bar{R}) \cup \{0\}$ and the proper closed sets of$Spec(\bar{R}) \cup \{0\}$ are in 1-1 correspondence with all closedsets of Spec(R). Thus, we have a bijection$Spec(\bar{R}) \cup_{Spec\,(T)} Spec(S^{-1}R) \longrightarrow Spec(R)$
which is both continuous and closed, therefore it is ahomeomorphism.
ii) By the universal property of pullback diagrams, R is alwaysidentified with a subring of $\tilde{R}$ via the injectiongiven by $\phi(r) = (\bar{r}, r/1)$.If R is saturated and $S^{-1}R \subset R^*$, we claim that ϕ must be surjectiveas well. To see this, let $\left(\bar{r}, \frac{a}{t}\right) \in \bar{R} \times_T S^{-1}R$ be arbitrary. By definition, $\bar{r} = \overline{(a/t)}$ in T, whence $b = r - \frac{a}{t} \in P(S^{-1}R)$. Therefore $P(S^{-1}R) = S^{-1}(P(R)) = P(R)$.

Thus, $b \in P(R) \subset R$. So $\frac{a}{t} = r - b \in R, and \left(\bar{r}, \frac{a}{t}\right) = \left(\left(\frac{\bar{a}}{t}\right), \frac{a}{t}\right) = \phi\left(\frac{a}{t}\right) \in \phi(R)$. □

Corollary 3.15.*If R is a Prüfer G-type domain, then $S^{-1}R \subset R^*$,R has pullback type, and $R^* = \cap \{R_P | P \in Spec^1(R)\}$has essential type. If in addition, R is a Bézout G-typedomain, then $S^{-1}R = R^*$.*

Proof. Since each Prüfer G-type domain is necessarilya GD-type domain (R as a G-type domain is called a going-downG-type domain if for every overring T of R, the inclusion map $R \longrightarrow T$ satisfies the going-down property), hence$S^{-1}R \subset R^*$ and obviously R has pullback type andfurthermore for each Prüfer domain (specially PrüferG-type domain) we have
$R^* = \cap \{R_P | P \in Spec^1(R)\}$ as a essential type .
Now if R is a *Bézout* G-type domain, since each overring of Ris a ring of fraction of R. So if $R^* = T^{-1}R$ for somesaturated multiplicatively closed set T, then:
$S^{-1}R \subset T^{-1}R = R^* = \cap \{R_P | P \in Spec^1(R)\}$.
Therefore,
$T \supset S$ and $R_P \supset T^{-1}R, \forall P \in Spec(R)$. It follows that $T \cap P = 0, \forall P \in Spec^1(R)$,and so $S \supset T$.Thus$S^{-1}R = T^{-1}R = R^*$. □

Remark 3.16. If *R* is a *G*-type domain such that *Spec(R)* is a countable set,then:
$S^{-1}R = \cap \{R_P | P \in Spec(R)\}$ is a countable set ofone-dimensional quasi local rings.The condition that $Spec^1(R)$be countable is characterized by $S^{-1}R$ being semiquasilocal ofdimension at most 1.
Therefore it is obvious that a G-type domain R satisfies Spec(R) is countable and R has essential type if and only ifevery nonzero principal ideal of R is countably intersection of (height 1) primary ideals.

Example 3.17. We exhibit a one-dimensional quasi local domain *R* such that $R^*$is a one-dimensional (therefore, essential) Prüfer *G*-typedomain, but not semi quasi local. Let *V* be a one-dimensionalvaluation domain with quotient field *K* such that there exists analgebraic field extension *L* of *K* having infinitely many valuation subrings extending *V*. (for instance, take $V = Z_{PZ}$ and L thefield of algebraic numbers.) Let *T* be the integral closure of *V* in*L*. Then *T* is one-dimensional and Prüfer, but notsemiquasilocal.

Corollary 3.18.*Let $R_1, R_2, \dots, R_n, \dots$ befinite-dimensional conducive domains which are not fields, withQ_i being the (unique) height 1 prime of R_i. Let$R = \cap_{i=1}^{\infty} R_i$ and pick$q_i = Q_i \cap R$. Let (V_i, M_i) be the (unique) one dimensional valuation overring of $R_i$and let $b_i = (R_i : V_i) \cap M_i$(which is nonzero by theconducive property). Let $W = \cap_{i=1}^{\infty} V_i$ and*
$m_i = M_i \cap W$. Assume further that R and each of the $R_i's$ have acommon quotient field K and that $q_i \nsubseteq q_j$ whenever$i \neq j$. then:
1) R is a G-type domain and $R^* = W$, a one-dimensionalsemi quasi local Bézout domain.
2) $Spec^1(R) = \{q_1, q_2, \dots, q_n, \dots\}$.
3) $S^{-1}R = \cap_{i=1}^{\infty} R_{q_i} \subset R^*$.
4) If each R_i is a one-dimensional, then $R = S^{-1}R$ isone-dimensional and semi quasi local.
5) If each R_i is saturated, then R is saturated and $R = \tilde{R} \simeq \bar{R} \times_{T^*} R^*$ where $T^*$is the total quotient ring of R^*/P^*.

Corollary 3.19.$Spec^1(R)$ is finite for every G-type domain R such that R' isa strong G-type domain.

Example 3.20. *Spec(R)* is finite and $dim(R) \leq 1$ if (and only if) R is acompactly packed *G*-type domain of essential type. (R is compactlypacked if, for any subset Ω of *Spec(R)* and any ideal I of$R$, the condition $I \subset \cup \{P | P \in \Omega\}$ implies$I \subset P$, $\exists$ P $\in \Omega$. In a compactly packedring, every prime ideal P is the radical of a principal ideal. By essentiality, $dim(R) \leq$ 1.Thus, for every P,$Spec(R) \backslash P$ is a quasi-compact Zariski- open set, andtherefore it is closed when *Spec(R)* is discrete. Since the patchtopology is compact, *Spec(R)* must be finite.

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