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A NOTE ON BENEDICKS' THEOREM ON HEISENBERG GROUP

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ABSTRACT

In this article we prove that if a integrable functions f on the Heisenberg group is supported on a 'thin' set and the group Fourier transform $\hat{f}(\lambda)$ *of the function is a rank one operator for all* $\lambda\neq 0$*, then the function is zero. This result generalizes the previously known result given in [7], where it was shown that if an integrable function f on the Heisenberg group is supported on the set* $B\times R$, $B\subset C^n$ *is of finite Lebesgue measure on* C^n*, and the group Fourier transform* $\hat{f}(\lambda)$ *is a rank one operator for all* $\lambda\neq 0$*, then f is the zero function.*

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1. Introduction

The uncertainty principle says that a nonzero function and its Fourier transform cannot both be sharply localized. There are several manifestations of this principle. We refer the reader to the excellent survey article by Folland and Sitaram [4] and by Havin and Jöricke [5].

In this paper we are interested in a variant of Benedicks' theorem on the Heisenberg group. In an interesting article [1], Benedicks has extended the classical Paley-Wiener theorem which is following. Let $f\in L^2(R^n)$. If both the sets $\{x\in R^n: f(x)\neq 0\}$ and $\{\xi\in R^n: \hat{f}(\xi)\neq 0\}$ have finite Lebesgue measure, then $f=0$. Later, a series of analogous results to the Benedicks theorem have been investigated in the context of noncommutative Lie groups, see [7, 8], for instance. In [7] Narayanan and Ratnakumar had worked out an analogous result to the Benedicks theorem for the partially compactly supported functions on the Heisenberg groups in terms of finite rank of the Fourier transform of the function. Further, in a recent article, M.K. Vemuri [10] has relaxed the compact support condition prescribed in [7] on function by finite Lebesgue measure. Since in the context of noncommutative Lie groups the Fourier transform is an operator valued function here they have measured the 'smallness' of the Fourier transform in terms of the rank of these operators.

In this article, we will prove a stronger result for certain class of functions. We will prove that if a function f defined on the Heisenberg group is supported in a 'thin set at infinity' and the group Fourier transform $\hat{f}(\lambda)$ is a rank one operator for each $\lambda\neq 0$, then the function f is the zero function. This 'thin set at infinity' includs the compact set as well as the sets of finite Lebesgue measure.

2 Heisenberg group

The n-dimensional Heisenberg group H^n is $C^n\times R$ equipped with the following group law;

$$(z,t).(w,s)=(z+w,t+s+\frac{1}{2}\zeta(z.\bar{w}))$$

where $\zeta(z)$ is the imaginary part of $z\in C$. Under this group law, H^n becomes a two step nilpotent Lie group with center $Z=\{0\}\times R$. For each $\lambda\in R\backslash\{0\}$, there exists an irreducible unitary representatrion π_λ acting on $L^2(R^n)$ given by

$$\pi_\lambda(z,t)\phi(\xi)=e^{it\lambda}e^{i\lambda(x.\xi+\frac{1}{2}x.y)}\phi(\xi+y),\ for\phi\in L^2(R^n) and\ \ z=x+iy\in C^n$$

These are the all infinite dimensional irreducible unitary representations of H^n up to unitary equivalence [3].

For $f\in L^1(H^n)$, we define the group Fourier transform $\hat{f}(\lambda)$, for $\lambda\neq 0$ by

$$\hat{f}(\lambda)=\int_{H^n} f(z,t)\,\pi_\lambda(z,t)\,dz\,dt.$$

Since $\pi_\lambda(z,t)$ is unitary it follows that $\hat{f}(\lambda)$ is a bounded operator on $L^2(R^n)$. Moreover, if $f\in L^1(H^n)\cap L^2(H^n)$, then $\hat{f}(\lambda)$ is Hilbert-Schmidt operator and we will denote it's Hilbert-Schmidt norm by $\left\|\hat{f}(\lambda)\right\|_{HS}$.

In the rest of this section, we recall the necessary details about the Weyl transform and the Fourier-Wigner transform. For a suitable function g defined on C^n. Weyl transform is defined to be the operator

$$W_\lambda(g)=\int_{C^n} g(z)\pi_\lambda(z)\,dz.$$

where $\pi_\lambda(z)=\pi_\lambda(z,0)$. Clearly $W_\lambda(g)$ defines a bounded operator on $L^2(R^n)$, if $g\in L^1(C^n)$. For $g\in L^2(C^n)$, $W_\lambda(g)$ is a Hilbert-Schmidt operator, and we have the Plancherel Theorem [9]:

$$\int_{C^n}|g(z)|^2\,dz=(2\pi|\lambda|)^{-n}\left\|w_\lambda(g)\right\|_{HS}^2$$

The twisted convolution of two functions F and G on C^n is defined to be

$$F\times_{\lambda}G(z)=\int_{C^n} F(z-w)G(w)e^{\frac{i\lambda}{2}\Im(z\cdot\bar{w})}\,dw.$$

It is known that $W_{\lambda}(F\times_{\lambda}G)=W_{\lambda}(F)W_{\lambda}(G)$. When $\lambda=1$, we write $F\times G$ instead of $F\times_1 G$ and call it the twisted convolution of F and G. Similarly $W_1(F)$ will be denoted by $W(F)$ and called the Weyl transform of F.

Let ϕ_1 and ϕ_2 belong to $L^2(R^n)$. The Fourier-Wigner transform of ϕ_1 and ϕ_2 is a function on C^n and is defined by

The Fourier-Wigner transform satisfies the 'orthogonality relation',

$$\int_{C^n} A(\phi_1,\phi_2)(z)\overline{A(\psi_1,\psi_2)(Z)}dz=(2\pi)^n<\phi_1,\psi_1><\phi_2,\psi_2> \qquad (1)$$

In fact, if $\{\phi_i:i\in N\}$ is an orthonormal basis for $L^2(R^n)$, then the collection $\{A(\phi_i,\phi_j):i,j\in N\}$ forms an orthonormal basis for $L^2(C^n)$ [10]. In particular, if $F\in L^2(C^n)$ is orthogonal to $A(\phi,\psi)$ for all $\phi,\psi\in L^2(R^n)$, then $F=0$.

3 Uncertainty principle for Fourier-Wigner transform

We start with the following definition as was defined in [2]. Let $A\subset R^n$ be given. We will say that A is a thin set at infinity if there is $R>0$ such that

$$\lim_{|x|\to\infty}\mu(A\cap B(x,R))=0$$

Here $\mu(X)$ denotes the Lebesgue measure of the set $X\subset R^n$. The following proposition says that every set of finite Lebesgue measure is a thin set at infinity; Consequently the class of thin sets at infinity is rich enough.

Proposition 1 (Prop. 3.3, [2]) *Let $A\subset R^n$ be given and let us consider the following conditions:*

1. *A has finite Lebesgue measure,*
2. *for almost every $x\in(0,1)^n$, the set $(x+Z^n)\cap A$ is finite (Benedicks condition),*
3. *A is a thin set at infinity. Then*

$$(1)\Rightarrow(2)\Rightarrow(3)$$

The purpose of this section is to show that all thin sets at infinity A are annihilating sets for the Fourier-Wigner transform, that is, if the support of $A(\phi_1,\phi_2)$ is contained in A (in the sense that $A(\phi_1,\phi_2)$ vanishes almost everywhere outside A), then either $\phi_1=0$ or $\phi_2=0$. This result is obtained in [2] for the short time Fourier transform. We restate their result in terms of the Fourier-Wigner transform.

Theorem 3.2 *Let $F(z)=A(\phi_1,\phi_2)(z)$, where $\phi_1,\phi_2 \in L^2(R^n)$. If the function F is supported inside a thin set at infinity A, then F=0.*

4. Uniqueness results for the group Fourier transform on the Heisenberg groups

Our main result is the following.

Theorem 4.1 *Let $f \in L^1(H^n)$ be supported on a set of the form $A\times R$, where $A \subset C^n$ is a thin set at infinity. If $\hat{f}(\lambda)$ is a rank one operator for all $\lambda \neq 0$, then $f=0$.*

Remark 4.2. *It follows from Proposition* 3.1 *that our result is stronger than that of [10], where it was assume that the Lebesgue measure of the set A is finite.*

Proof of Theorem 4.1. Let $f^\lambda(z)$ denote the partial Fourier transform of *f* in the *t*-variable. That is,

$$f^\lambda(z)=\int_R f(z,t)e^{i\lambda t}\,dt.$$

Then a simple computation shows that $\hat{f}(\lambda)=W_\lambda(f^\lambda)$. By the hypothesis we have that $f^\lambda(z)$ is supported in the set *A* and $\hat{f}(\lambda)=W_\lambda(f^\lambda)$ is a rank one operator for all $\lambda \neq 0$. As in [?], it is enough to prove $f^\lambda=0$, for $\lambda=1$. It suffices to show that if $F\in L^1(C^n)$ is supported on *A* and $W(F)$ is a rank one operator, then $F=0$. This immediately follows from Theorem 3.2 once we show that *F* is the Fourier-Wigner transform of two functions in $L^2(R^n)$. For this, $\overline{G}=F$.Since is a rank one operator, we have $\psi_1,\psi_2 \in L^2(R^n)$ such that

$$W(\overline{G})\phi = <\phi,\psi_1>\psi_2,\ for \psi_1,\psi_2 \in L^2(R^n)$$

Hence, if $\psi \in L^2(R^n)$ we have

$$\begin{aligned} <W(\overline{G})\phi,\psi> &= \int_{C^n} \overline{G}(z) <\pi_1(z)\phi,\psi> dz \\ &= \langle \phi,\psi_1\rangle \langle \psi_2,\psi\rangle \\ &= (2\pi)^{-n} \int_{C^n} A(\phi,\psi)(z)\ \overline{A(\psi_1,\psi_2)(z)}\ dz \end{aligned}$$

Here the last step follows from2.1. Therefore from the above inequalities follows that $G(z)=A(\psi_1,\psi_2)(z)$ and by Theorem3.2, it follows that $G=0$ and hence so is *F*.

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