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MINIMIZATION IN GENERATING SPACE AND FIXED POINT

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ABSTRACT

A non-convex minimization theorem has been established for generating space of quasi 2-metric family for sequence of mappings with non commuting weak compatible condition. Also supported by an example.

KEY WORDS: Generating space of quasi 2-metric family, weak compatible mapping, Minimization theorem, common fixed point.

Mathematics Subject Classification: 47H10, 54H25.

1. **INTRODUCTION**

An important area of fixed point theory is the generating space of quasi 2-metric family, because of its involvement and application to fuzzy and probabilistic 2-metric space and a minimization theorem [1], [3] is to obtain fixed point theorem. In 2008 V. B. Dhagat and V. S. Thakur [2] proved non convex minimization theorem for generating space of quasi 2-metric family. In this paper we prove a minimization theorem for sequence of mappings T^{a} for $a \in N$ and further we prove fixed point theorem as an application of minimization theorem with non commuting condition known as weak compatible.

2. **PRELIMINARIES**

2.1 **Generating space of quasi 2-metric family:-**

Generating space of quasi 2-metric family already defined[1] and [2] as follows:-

Let X be a non empty set and $\{D_\alpha : \alpha \in (0,1]\}$ be family of mapping D_α from $X \times X \times X$ into R^+. $\{X, D_\alpha\}$ is called generating space of quasi 2-metric family if it satisfy following axioms:

(GM 1) – For any two distinct points x and y there exit z in X such that

$$D_\alpha(x, y, z) \neq \alpha \in (0,1]$$

(GM 2) – $D_\alpha(x, y, z) = 0$ if at least two x, y, z are equal and $\alpha \in (0,1]$

(GM 3) – $D_\alpha(x, y, z) = D_\alpha(x, z, y) D_\alpha(z, y, x) = \cdots$ ... ... for all x, y, z in X and $\alpha \in (0,1]$

(GM 4) – for any $\alpha \in (0,1]$ there exists $\alpha_1, \alpha_2, \alpha_3, \in (0, \alpha]$ such that $\alpha_1 + \alpha_2 + \alpha_3, \leq (0, \alpha]$ and so $D_\alpha(x, y, z) \leq D_{\alpha_1}(x, y, u) + D_{\alpha_2}(x, u, z) + D_{\alpha_3}(u, y, z)$

(GM 5) – $D_\alpha(x, y, z)$ is non increasing and left continuous in α and $\forall\, x, y, z \; in \; X$. Throught this paper, we assume that $k: (0,1] \to (0, \infty)$ is non decreasing function satisfying the condition

$$K = Sup\; k(\alpha)$$

Let E and F be mappings from generating space of quasi 2-metric family $\{X, D_\alpha\}$ into itself. The mapping E and F are said to be weak compatible if it commute at convergent point. i.e. for sequence x_n in X such that

$\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = t$ for some t in X then $EFt = FEt$.

3. MAIN RESULT

Theorem 3.1. Let $\{X, D_\alpha : \alpha \in (0,1]\}$ and $\{Y, D'_\alpha : \alpha \in (0,1]\}$ be two complete generating space of quasi 2-metric family. $f: X \to Y$ be a closed and $T^a: X \to X$ be continuous mapping satisfying for all $\alpha \in N$

(i) $D_\alpha(T^a x, T^a y, z) \le \max\{D_\alpha(T^a x, y, z).(x, T^a y, z).(x, y, T^a z)\}$ and

(ii) $D'_\alpha\big(f(T^a x).f(T^a y).f(z)\big)$

$$\le max\{D'_\alpha\big(f(T^a x).f(y).f(z)\big), D'_\alpha\big(f(x).f(T^a y).f(z)\big), D'_\alpha\big(f(x).f(y).f(T^a z)\big)\},$$

$\forall\ x, y, z \in X$ and $\alpha \in (0,1]$

(iii) $\Psi: \Re \to \Re$ be non decreasing continuous and bounded below function,

(iv) $\emptyset: f(x) \to \Re$ be a lower semi continuous and bounded below function,

(v) $for\ any\ p \in X$ with $\inf \Psi\Big(\emptyset\big(\mathrm{f}(x)\big)\Big) < \Psi\Big(\emptyset\big(\mathrm{f}(p)\big)\Big)$ there exists q with $p \ne Tq$ and

$\max[max\{D_\alpha(T^a, q, p, z), D_\alpha(q, T^a p, z), D_\alpha(q, p, T^a z)\}],$

$c.max\{D'_\alpha\big(f(T^a q).f(p).f(z)\big), D'_\alpha\big(f(q).f(T^a p).f(z)\big), D'_\alpha\big(f(q).f(p).f(T^a z)\big)\}$

$\le K(\alpha)\Big[\Psi\Big(\emptyset\big(\mathrm{f}(p)\big)\Big) - \Psi\Big(\emptyset\big(\mathrm{f}(q)\big)\Big)\Big]$ $\forall\ x, y, z \in X$ and $\alpha \in (0,1]$

And c is any constant.

Then there exists an x_0 in$X$ such that with $\inf \Psi\Big(\emptyset\big(\mathrm{f}(x)\big)\Big) = \Psi\Big(\emptyset\big(\mathrm{f}(p)\big)\Big)$.

Proof: Let us suppose $\inf \Psi\Big(\emptyset\big(\mathrm{f}(x)\big)\Big) < \Psi\Big(\emptyset\big(\mathrm{f}(p)\big)\Big)$ for every y in X and choose $r \in X$

For which $\inf \Psi\Big(\emptyset\big(\mathrm{f}(r)\big)\Big)$ is defined then inductively we define a sequence $\{r_n\} \subset X$ with $r_1 = r$. suppose r_n is know is consider

$W_n =$

$\left\{\left[w \in\right.\right.$

X:max $maxDαTaw,$ $rn,z,Dαw,$ $Tarn,z,Dαw,$
$rn,Taz,c.maxD'αfTaw.frn.fz,D'αfw.fTarn.fz,D'αfw.frn.fTaz$

$\leq K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) - \Psi\left(\emptyset\left(\mathrm{f}(w)\right)\right)\right] \ \forall\, x, y, z \in X$ and $\alpha \in (0,1]$

W_n is non empty set and there exists $w \in W_n$ such that $r_n \neq Tw$. We can choose $r_{n+1} \in W_n$ such that

$r_n \neq T(r_{n+1})$ and

$$\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) \leq inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) + 1/3\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) - inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right)\right].$$

Clearly $\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right)$ is a non increasing lower bounded sequence. Hence it is a convergent sequence.

Now we prove $\{r_n\}$ and $\{(r_n)\}$are Cauchy sequences:

$$max\left\{D_\alpha(T^a r_n, T^a r_{n+1}, w), D'_\alpha\left(f(T^a r_n).f(r_{n+1}).f(w)\right)\right\}$$

$$\leq$$

$$max\begin{bmatrix} max\{D_\alpha(f(T^a r_n), r_{n+1}, w), D_\alpha(r_n, T^a r_{n+1}, w), D_\alpha(r_n, r_{n+1}, T^a w)\}, \\ c.max\left\{D'_\alpha\left(f(T^a r_n).f(r_{n+1}).f(w)\right), D'_\alpha\left(f(r_n).f(T^a r_{n+1}).f(w)\right), D'_\alpha\left(f(r_n).f(r_{n+1}).f(T^a w)\right)\right\}\end{bmatrix}$$

$$\leq K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) \leq inf\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right)\right]$$

$\forall\ n, m \in N, n < m \Longrightarrow$ there exists $\alpha_j = \alpha_j(n,m);\ \sum \alpha_j \leq \alpha$, such that

$$max\left\{\begin{matrix} max\left\{D_{\alpha_j}(T^a r_n, r_m, w), D_{\alpha_j}(r_n, T^a r_m, w), D_{\alpha_j}(r_n, r_m, T^a w)\right\}, \\ c.max\left\{D'_{\alpha_j}\left(f(T^a r_n).f(r_m).f(w)\right), D'_{\alpha_j}\left(f(r_n).f(T^a r_m).f(w)\right), D'_{\alpha_j}\left(f(r_n).f(r_m).f(T^a w)\right)\right\}\end{matrix}\right\}$$

$\leq$

$$\sum_{j=n} max \begin{cases} max\left\{D_{\alpha_j}(T^a r_n, r_m, w), D_{\alpha_j}\left(r_j, T^a r_{j+1}, w\right), D_{\alpha_j}\left(r_j, r_{j+1}, T^a w\right)\right\}, \\ c.max\left\{D'_{\alpha_j}\left(f\left(T^a r_j\right).f\left(r_{j+1}\right).f(w)\right), D'_{\alpha_j}\left(f\left(r_j\right).f\left(T^a r_{j+1}\right).f(w)\right), D'_{\alpha_j}\left(f\left(r_j\right).f\left(r_{j+1}\right).f(T^a w\right.\right. \end{cases}$$

Hence, $\forall\, n, m \in N, n < m;$

$\leq$

$$max\begin{bmatrix} max\{D_\alpha(T^a r_n, r_m, w), D_\alpha(r_n, T^a r_m, w), D_\alpha(r_n, r_m, T^a w)\}, \\ c.max\{D'_\alpha\left(f(T^a r_n).f(r_m).f(w)\right), D'_\alpha\left(f(r_n).f(T^a r_m).f(w)\right), D'_\alpha\left(f(r_n).f(r_m).f(T^a w)\right)\} \end{bmatrix}$$

$$\leq K(\mu)\sum_{j=n}^{m-1}\left[\Psi\left(\emptyset\left(\mathrm{f}\left(r_j\right)\right)\right) - inf\Psi\left(\emptyset\left(\mathrm{f}\left(r_{j+1}\right)\right)\right)\right]$$

$$\leq K(\alpha)\sum_{j=n}^{m-1}\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) - inf\Psi\left(\emptyset\left(\mathrm{f}(r_m)\right)\right)\right]$$

For some α_j with $0 < \alpha_{j+1} < \alpha_k \leq \alpha\ j = n \ldots\ldots\ldots\ldots. m - 1$

$$D_\alpha(r_n, r_{n+1}, w) \leq D_{\alpha_1}(r_n, r_{n+1}, T^a r_{n+1}) + D_{\alpha_2}(r_n, T^a r_{n+1}, w) + D_{\alpha_3}(T^a r_{n+1}, r_{n+1}, w)$$

$\leq$

$$D_{\alpha_1}(r_n, r_{n+1}, T^a r_{n+1}) + D_{\alpha_2}(r_n, T^a r_{n+1}, w) + D_{\alpha_3}(T^a r_{n+1}, r_{n+1}, T^a r_n) +$$
$$D_{\alpha_4}(T^a r_{n+1}, T^a r_n, w) + D_{\alpha_5}(T^a r_{n+1}, r_{n+1}, w)$$

For $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \leq \alpha$

$\leq$

$$3\begin{bmatrix} max\{D_\alpha\left((T^a r_n), r_{n+1}, w\right), D_\alpha(r_n, T^a r_{n+1}, w), D_\alpha(r_n, r_{n+1}, T^a w)\}, \\ c.max\{D'_\alpha\left((T^a r_n).f(r_{n+1}).f(w)\right), D'_\alpha\left(f(r_n).f(T^a r_{n+1}).f(w)\right), D'_\alpha\left(f(r_n).f(r_{n+1}).f(T^a w)\right)\} \end{bmatrix}$$

$$\leq 3K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) - inf\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right)\right]$$

Then also we get

$$D_\alpha(r_n, r_{n+1}, w) \leq 3K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) - inf\Psi\left(\emptyset\left(\mathrm{f}(r_m)\right)\right)\right]$$

Where $n < m$

In the manner we obtain

$$D'_{\alpha}\left(f(r_n).f(r_{n+1}).f(w)\right) \le 3K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) - inf\Psi\left(\emptyset\left(\mathrm{f}(r_m)\right)\right)\right]$$

Where $n < m$

Hence $\{r_n\}$ and $\{f(r_n)\}$ are Cauchy sequences.

Assume that $\lim_{n\to\infty} r_n = A$ and $\lim_{n\to\infty} f(r_n) = B$.

Since f is closed therefore $f(A) = B$.

By the continuity of Ψ and lower semi continuity of $\emptyset$ we have

$$\Psi\left(\emptyset\left(\mathrm{f}(b)\right)\right) \le \lim_{n\to\infty}\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) = \lim_{n\to\infty}\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right)$$

Let $\delta = inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) \in \mathrm{R}$

$\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right) \le inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) + 1/3\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) - inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right)\right]$, we have

$$\lim_{n\to\infty}\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right) \le \left({}^{2}/_{3}\right)\delta + \frac{1}{3\lim_{n\to\infty}\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right)} =$$

$$\left({}^{2}/_{3}\right)\delta + 1/3\lim_{n\to\infty}\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right)$$

Which is contraction, therefore there exists x_0 in X such that

$$inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) = \emptyset\left(\mathrm{f}(\,x_0)\right)$$

Now we give a fixed point theorem as an application of the above theorem under non commuting condition known as weak compatible.

Theorem 3.2 Let $\{X, D_\alpha : \alpha \in (0,1]\}$ and $\{Y, D'_\alpha : \alpha \in (0,1]\}$ be two complete generating space of quasi 2-metric family. $f: X \to Y$ be a closed and $T^a, S^a: X \to X$ be continuous mapping satisfying

(i) $D_\alpha(T^a x, T^a y, z) \leq \max\{D_\alpha(T^a x, y, z).(x, T^a y, z).(x, y, T^a z)\}$ and

(ii) $D'_\alpha\left(f(T^a x).f(T^a y), f(z)\right)$

$$\leq max\left\{D'_\alpha\left(f(T^a x).f(y).f(z)\right), D'_\alpha\left(f(x).f(T^a y).f(z)\right), D'_\alpha\left(f(x).f(y).f(T^a z)\right)\right\},$$

$\forall\ x, y, z \in X$ and $\alpha \in (0,1]$

(iii) $\Psi: \mathfrak{R} \to \mathfrak{R}$ be non decreasing continuous and bounded below function,

(iv) $\emptyset: f(x) \to \mathfrak{R}$ be a lower semi continuous and bounded below function,

(v) S^a and T^a are weak compatible and

max[$max\{D_\alpha(T^a, T^a S^a x, z), D_\alpha(x, T^a S^a x, z), D_\alpha(x, S^a x, T^a z)\}$],

$$c.max\left\{D'_\alpha\left(f(T^a x).f(T^a S^a x).f(z)\right), D'_\alpha\left(f(x).f(T^a S^a x).f(z)\right), D'_\alpha\left(f(x).f(S^a x).f(T^a z)\right)\right\}$$

$\leq K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) - \Psi\left(\emptyset\left(\mathrm{f}(S^a x)\right)\right)\right]\ \forall\ x, y, z \in X$ and $\alpha \in (0,1]$

And c is any constant. Then there exists unique common fixed point x_0 in X.

Proof: If $x_0 \in X$ such that $inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) = \Psi\left(\emptyset\left(\mathrm{f}(x_0)\right)\right)$

then $x_0 = T^a S^a x_0 . S^a x_0 = T^a x_0$ therefore some $\alpha \in (0,1]$

$0 < max\{D_\alpha(T^a, T^a S^a x, z), D_\alpha(x, T^a S^a x, z), D_\alpha(x, S^a x, T^a z)\}$

$\leq K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(x_0)\right)\right) = \Psi\left(\emptyset\left(\mathrm{f}(S^a x_0)\right)\right)\right] \leq 0$

$which\ is\ contraction.$ then $Sx_0 = Tx_0$.

Now by weak compatible of $T^a\ and\ S^a$

$S^a x_0 = T^a S^a x_0 = S^a T^a x_0 = T^a x_0$.

Also for some $\alpha_1, \alpha_2, \alpha_3 \in (0,1]$ such that $\alpha_1 + \alpha_2 + \alpha_3 \leq \alpha$

$$D_\alpha(x_0, T^a x_0, z) \leq D_{\alpha_1}(x_0, T^a x_0, T^a S^a x_0) + D_{\alpha_2}(x_0, T^a S^a x_0, z) + D_{\alpha_3}(T^a S^a x_0, T^a x_0, T^a x_0, z)$$

$\leq D_{\alpha_3}(T^a S^a x_0, T^a x_0, T^a x_0, z) = 0.$ hence $T^a x_0 = S^a x_0 = x_0$

$\boldsymbol{uniqueness}$: Let us assume there exists another fixed point y_0 such that

$S^a y_0 = T^a y_0 = y_0$ and by **theorem 3.1** we have $inf\Psi\left(\emptyset\big(\mathrm{f}(x)\big)\right) = \emptyset\big(\mathrm{f}(\,y_0)\big)$.

But $inf\Psi\left(\emptyset\big(\mathrm{f}(x)\big)\right) = \emptyset\big(\mathrm{f}(\,x_0)\big)$ hence by uniqueness of infima we get $x_0 = y_0$

Remark: Theorem 3.1 and 3.2 can be proved easily for convergent sequence of mappings.

Corollary: Let $\{X, D_\alpha : \alpha \in (0,1]\}$ and $\{Y, D'_\alpha : \alpha \in (0,1]\}$ be two complete generating space of quasi 2-metric family. $f : X \to Y$ be a closed, $\emptyset : f(X) \to \Re$ be a lower semi continuous and bounded below function. Let $S^a : X \to X$ be a mapping such that $\forall\, x, y, z \in X$ and c is any continuous mapping satisfying

$\max\{D_\alpha(S^a x, x, z). D'_\alpha\big(f(S^a x), f(x), f(z)\big)\}$

$\leq K(\alpha)[\emptyset(\mathrm{x}) = \emptyset(S^a x)]$

Proof: Consider $T = 1$ and $\Psi = 1$ we get required result.

Example:

Let $X = [0,1]$ $Y = [0, \infty]$, $D_\alpha = D'_\alpha = D_1$ defined by $D_1(x, y, z) = \frac{D(x,y,z)}{1+D(x,y,z)}$

And $D(x, y, z) = max\{|x - y| + |y - z| + |z - x|\}$,

The mapping defined as follows:

$T^a : X \to X$ as $T^a x = x^{2a}$ $\;f : X \to X$ as $fx = x$, $\emptyset : f(x) \to R$ as $\emptyset(x) = 1/(1 - x)$

and $\Psi : \mathrm{R} \to \mathrm{R}$ $\Psi(\mathrm{x}) = x^2/2$ and $K(\alpha) = 3$ satisfy the all conditions of theorem 3.1.

$also$ $S^a : X \to X$ is defined $S^a x = \frac{x^{2a}}{2a}$, then (S, T) is weak compatible which satisfying the condition of theorem 3.2, hence 0 is a unique fixed point.

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