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SECOND-ORDER CONE PROGRAMMING

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ABSTRACT

SOCPs are non-linear convex optimizations. Linear program (LP), quadratic program (QP) and quadratically constrained quadratic programs (QCQP) can all be formulating as SOCP problems. This paper tried to discuss how to formulate such problem as SOCPs. There are many variants of interior-point methods such as projective or potential reduction, affine, path- following and logarithmic barrier methods. Among these methods, we focused on logarithmic barrier methods for solving SOCP.

Key Words: Second Order Cone Programming (SOCP), Logarithmic Barrier Method, combinatorial optimization. Interior-point (IP), and Convex Optimization

1. Background of Study

Second-order cone programming (SOCP) problems are convex optimization problems in which a linear function is minimized over the intersection of an affine linear manifold with the Cartesian product of second-order (Lorentz) cones. Linear programs, convex quadratic programs and quadratically constrained convex quadratic programs can all be formulated as SOCP problems, as can many other problems that do not fall into these three categories. These SOCP problems model applications from a broad range of fields from engineering, control and finance to robust optimization and combinatorialoptimization.

SOCP is a special class of semi-definite programming. Semi-definite programming (SDP) is the optimization problem over the intersection of an affine set and the cone of positive semi-definite matrices. Therefore, SOCP is less general than semi-definite programming.SOCP falls between linear programming (LP) and quadratic programming (QP). Like LP and QP problems, SOCP problems can be solved by interior point methods. The computational effort per iteration required by these methods to solve SOCP problems is greater than that required to solve LP and QP problems but less than that required to solve SDP's of similar size and structure. The set of feasible solutions for SOCP problem is not polyhedral as it is for LP and QP problems.Interior-point (IP) has dominated the research on convex optimization from the early 1990s until recently. They are popular because they reach a high accuracy in a small number of iterations, almost independent of problem size, type and data. Each iteration requires the solution of a set of linear equations with fixed dimensions and known structure. As a result, the time needed to solve different instances of a given problem family can be estimated quite accurately.

IP methods depend on only a small number of algorithm parameters, which can be set to values that work well for a wide range of data, and do not need to be tuned for a specific problem. The key to efficiency of an IP solver is the set of linear equations solved in each iteration. These equations are sometimes called Newton equations, because they can be interpreted as a linearization of the nonlinear equations that characterize the central path, or Ka rush-Kuhn-Tucker (KKT) equations, because they can be interpreted as optimality (or KKT) conditions of an equality-constrained quadratic optimization problem. The cost of solving the Newton equations determines the size of the problems that can be solved by an IP method

One of the advantages of IP methods is that they can easily be extended from the LP case to other optimization problems such as second-order cone programming and semi-definite programming. In this seminar the interior-point frame work will be introduced by applying a modification of Newton's method on the KKT conditions and explains how we can use logarithmic barrier method to approximate the optimal solutions of theproblem.

2. MATHEMATICALPRELIMINARIES

In this chapter we deal with some basic definitions and theorems that are required for the discussion of the main body of the journal

Definition 2.1

The convex optimization problem in standard form:

Minimize$f_0(x)$ (2.1)

Subject to: $f_j(x) \leq 0, j = 1, \ldots, n$

$h_i(x) = a_i{}^T x\text{-}b_i = 0, i = 1, \ldots, q$,where $x \in R^n$ is called a convex optimization problem if:

(i) The objective function $f_0$is convex,

(ii) Thefunctionsdefiningtheinequalityconstraints,$f_j, j = 1, \ldots, n$ areconvex,

(iii) the functions defining the equality constraints, $h_i = 0$ $i = 1, \dots, q$ are affine.

Definition 2.2

Lagrangian function $L: R^n \times R^m \times R^q \rightarrow R$ associated with equation (2.1)

$L(x, y, s) = f_0(x) + \sum_{j=1}^{m} y_j f_j(x) + \sum_{i=1}^{q} s_i h_i(x)$ (2.2)

where y_j and s_i are Lagrangian multipliers.

I. KARUSH-KUHN-TUCKER CONDITIONS

Considerthe convexoptimization problem (2.1), thefunctions: $f_0: S \rightarrow R$, $f_j: S \rightarrow R^m$ and $h_i: S \rightarrow q$ are all differentiable on some open set $S \rightarrow R^n$. The Karush-Kuhn-Tucker conditions become:

$$\nabla f_0(x) + \sum_{j=1}^{m} \left(y_j \nabla f_j(x)\right) + \sum_{i=1}^{q} s_i \nabla h_i(x)) = 0 \quad (2.3)$$

For $y_j f_j(x) = 0, j = 1, \dots, m$

$f_j \leq 0, j = 1, \dots, m$ b

$h_i = 0, i = 1, \dots, q$ b

$y_j \geq 0$, $j = 1, \dots, m$

where y_j and s_i denote the Lagrange multipliers associated with the constraints $f_j \leq 0$ and $h_i = 0$ respectively.

Definition2.3

The Jacobean matrix is the matrix of all first-order of partial derivatives of a vector-valued function. Suppose $\boldsymbol{f}: R^n \rightarrow R^m$ is a function which takes as input thevector $x \in R^n$ and products as output the vector $\boldsymbol{f}(x) \in R^m$.Then, the Jacobian matrix J of $\boldsymbol{f}$ is a $\boldsymbol{n} \times n$ matrix usually defined and arranged as follows:

$$J = \frac{df}{dx} = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (2.4)$$

Definition2.4:

The Hessian matrix is a square matrix of second–order partial derivatives of a scalar-valued function. Specifically, suppose $f: R^n \rightarrow R$ is a function taking as input a vector $x \in R^n$ and out putting a scalar $f(x) \in R$. If all second partial derivatives of f exist and are continuous over the domain of the function, Then the Hessian matrix $H(x)$ of f is square $n \times n$ matrix, usually defined and arranged as follows.

$$H(x) = \begin{bmatrix} \frac{\partial^2 f_1}{\partial {x_1}^2} & \cdots & \frac{\partial^2 f_1}{\partial x_1 \partial x_n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{\partial^2 f_n}{\partial x_n x_1} & \cdot \; \cdot \; \cdot & \frac{\partial^2 f_n}{\partial^2 {x_n}^2} \end{bmatrix}$$

Theorem: The first order necessary conditions for optimality

Let $f: R^n \to R$be differentiable at a point$\bar{x} c R^n$. If $\bar{x}$is a local solution to the problem P,then $\nabla f(\bar{x}) = 0$.

Proof: From the definition of the derivative we have that
$f(x) = f(\bar{x}) + \nabla f(\bar{x})T(x - \bar{x}) + 0(||x - \bar{x}||)$where

$$\lim_{x \to \bar{x}} \frac{o(\|x - \bar{x}\|)}{\|x - \bar{x}\|} = 0$$

Let $x = \bar{x} - t\nabla f(\bar{x})$ forsufficiently small $t > 0$.Then

$$0 \le \frac{f\big(\bar{x} - t\nabla f(\bar{x}) - f(\bar{x})\big)}{t} = -\|\nabla f(\bar{x})\|^2 + \frac{o(t)\|\nabla f(\bar{x})\|}{t}$$

Taking the limit as $t \to 0$,we obtain$\|\nabla f(\bar{x})\|^2 \le 0$

Therefore $\|\nabla f(\bar{x})\| = 0$.

Hence, $\nabla f(\bar{x}) = 0$. $\bar{x}$ is a stationary point if $\nabla f(\bar{x}) =$

Definition2.5:

A set$S \subseteq R^n$is a convex set if f it contains the line segment joining any of its points. i.e., $x, y \in S$ and $h \in [0,1]$such that $hx + (1 - h)y \in S$.

Example: $C = \{(x1, x2, x3) \in R^3$: $x_1 + 2x_2 - x_3 = 4\}$

Definition2.6:

A function$f: S \to \mathbb{R}$definedonaconvexsubset S ofR^nisconvexifforany $x, y \in S$and $h \in [0,1]$, we have $f(hx + (1 - h)y) \le hf(x) + (1 - h)f(y)$.If$f$ is twice continuously differentiable and the domain is there alien, then$f$isconvex.
$i.e., f''(x) \ge 0, \forall x$intheinterval,then$f$isconvex.

Proposition 2.6: Let $f_i: R^n \to R, i = 1, \dots, n$be given functions, let $h_1, \dots, h_n$be positive scalars, and consider the function$g: R^n \to R$,given by:

$g(x) = h_1 f_1 + \cdots + h_n f_n(x)$,if $f_1, \dots, f_n$are convex, then$g$ is also convex.

Proof: Let$f_1, \dots, f_n$be convex. We use the definition of convexity to show that g is convex. Let $x, y \in R^n$and$\alpha \in [0,1]$

$$g(\alpha x+(1-\alpha)y)=\sum_{j=1}^{m}(\lambda_j f_j(\alpha x+(1-\alpha)y \le \sum_{j=1}^{m}(\lambda_j(\alpha f_j(x)+(1-\alpha)f_j(y))$$
$$=\sum_{j=1}^{m}\lambda_j f_j(x)+(1-\alpha)\sum_{j=1}^{m}\lambda_j f_j(y)=\alpha g(x)+(1-\alpha)g(y)$$

Hence g is convex.

Definition 2.7:

A set $S \subseteq R^n$ is affine if the line through any two distinct points in S lies in S i.e., if for any x_1, x_2 and $h \in \mathbb{R}$, we have $hx_1 + (1-h)x_2 \in S$. Inother words, S contains the linear combination of any two points in S, provided the coefficients in the linear combination sum to one. An affine set contains every affine combination of its points: if S is affine set, $x_1, \ldots, x_k \in S$, and $h_1 + \cdots + h_k = 1$, then the point $h_1x_1 + \cdots + h_kx_k$ also belongs to S.

Example: solution set of linear equations$\{x: Ax = b\}$,every affine set can be expressed as solution set of system of linear equation.

Definition 2.8: A set K is called a cone if $\theta x \in K$ for each $x \in K$ and $\theta \ge 0$.

Examples: $\mathbb{R}+ = \{x \in \mathbb{R}: x \ge 0\}$, $k = \left\{x \in R^n: \sum_{i=1}^{q} {x_i}^2, x_1 \ge 0\right\}$second-order cone)

Definition 2.9:

A convex cone K is a cone with additional property that $x + y \in K$ for each $x, y \in K$ or for any $x, y \in K$ and $\theta_1, \theta_2 \ge 0$, we have$\theta_1 1 + \theta_2 x_2 \in K$.

Definition 2.10: A cone $K \subseteq R^n$ is called a proper cone if it satisfies the following:

(i) K is closed, $x, y \in K \Rightarrow x + y \in K$,

(ii) K is pointed cone with property that$K \cap (-K) = \{0\}$,which means that if it does not contain any subspace except the origin.

(iii) Convex and full-dimensional cone $(i.e. dim(K) = n)$. A full-dimensional cone is a cone which contains n linearly independent vectors.

Definition 2.11 (dual cone): The dual cone $K^*$of a proper cone is the set$\{s: s^T x \ge 0, \forall x \in K\}$ The dual cone satisfies several properties, such as:

- ✓ $K^*$is closed and convex
- ✓ $K_1 \subseteq K_2$implies ${K_1}^* \subseteq {K_2}^*$
- ✓ If K has nonempty interior, then K^* is pointed
- ✓ K^{**}= K, if K is convex and closed.

Definition 2.12: A function $f: R^n \to R$ with domain $f = R^n$ is called a norm if

(i) f is nonnegative: $f(x) \geq 0$ for all $x \in R^n$.

(ii) f is definite: $f(x) = 0$ if and only if $x = 0$.

(iii) f is homogeneous: $f(cx) = |c|f(x)$, for all $x \in R^n$ and $c \in \mathbb{R}$

(iv) f satisfies the triangle inequality: $f(x+y) \leq f(x) + f(y)$, for all $x, y \in R^n$.

We use the notation $f(x) = ||x||$, which is meant to suggest that a norm is a generalization of the absolute value on $\mathbb{R}$.

Definition2.13: The standard Euclidean normis:

$||x|| = \sqrt{xx^T} = (\sum_{i=1}^{q} x_i{}^2)^{\frac{1}{2}}$ where $x \in R^n$

Cauchy-Schwarz inequality: state that $x^T s \leq ||x|| \, ||s||$ for $x, s \in R^n$ the inequality holds with equality if and only if x and s are linear dependent.

Definition 2.14: A second order cone of dimension n is defined as

$k = \left\{ \begin{bmatrix} x_0 \\ x \end{bmatrix} : x_0 \in R, x \in R^{n-1} \right\}$ for $||x|| \leq x_0, x_0 \geq 0$

A second order cone is also called quadratic or Lorentz cone. For $n = 1$ we define the unit second order cone as:

$K_1 = \{x : x \in \mathbb{R}, x \geq 0\}$

Note that the second-order cone K is a convex set in R^n because for any two points in K, $[x_{01} \quad x_1{}^T]^T$ and $[x_{02} \quad x_2{}^T]^T$, and $\lambda \in [0,1]$, we have:

$$\lambda \begin{bmatrix} x_{01} \\ x_1 \end{bmatrix} + (1-\lambda) \begin{bmatrix} x_{02} \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_{01} + (1-\lambda) x_{02} \\ x_1 + (1-\lambda) x_2 \end{bmatrix} \in K$$

as $\lambda x_{01} + (1-\lambda) x_{02} \in R$, $\lambda x_1 + (1-\lambda) x_2 \in R^{n-1}$ and $||\lambda x_1 + (1-\lambda) x_2|| \leq \lambda ||x_1|| + (1-\lambda)\,)||x_2|| \leq \lambda x_{01} + (1-\lambda) x_{02}$.

3. SECOND ORDER CONEPROGRAMMING

Second order cone programming is a generalization of linear and quadratic programming that allows for affine combination of variables to be constrained inside a special convex set, called second order cone. The SOCP problem includes LPs, QPs and QCQPs as special classes with convex objective function and constraints of affine set. In this chapter we deal with SOCP, QP, QCQP, LP problems and approximate theirsolution.

3.1. SECOND-ORDER CONEPROGRAMMING

The primal second-order cone-programming (SOCP) problem is a constrained optimization problem that can be formulated as:

Minimize $\sum_{i=1}^{q} c_i^T x_i$ (3.1a)

Subject to $\sum_{i=1}^{q} \hat{A}_i x_i = b$ (3.1b)

$x_i \in K_i$ for $i = 1,2,\dots,q.$ (3.1c)

where $x_i \in R^{n_i \times 1}$is a variable, $A_i \in R^{m \times n_i}$is a given matrix, $n > m$,$\hat{c}_i \in R^{n_i \times 1}$and $b \in R^{m \times 1}$are given vectors, and$K$is the Cartesian product of second-order cones: $K = k_1 \times k_2 + \cdots + k_q$,and $k_i$is an n_i dimensional second order cone which is defined by:

$K_i = \{(x_0, x)^T \in R^{n_i} : x_0 \geq \|x\|\}$,$x_0 \in R, x \in R^{n_i - 1}, \|x\| = \sqrt{{x_1}^2 + {x_2}^2 + \cdots + {x_q}^2}$,$n = \sum_{i=1}^{q} n_i$

It is interesting to note that the SOCP problems in Eq.(3.1) are involving a linear objective function and a linear equality constraint. Each variable vector $x_i$in an SOCP problem is constrained to the second-order coneK_i. For $n = 1$, the second-order cone degenerates into a ray on the x-axis starting from $x = 0$, as shown in Fig. (3.1a). The second-order cones for $n = 2$ and 3 are depicted in Fig.3.1b and c, respectively as follows.

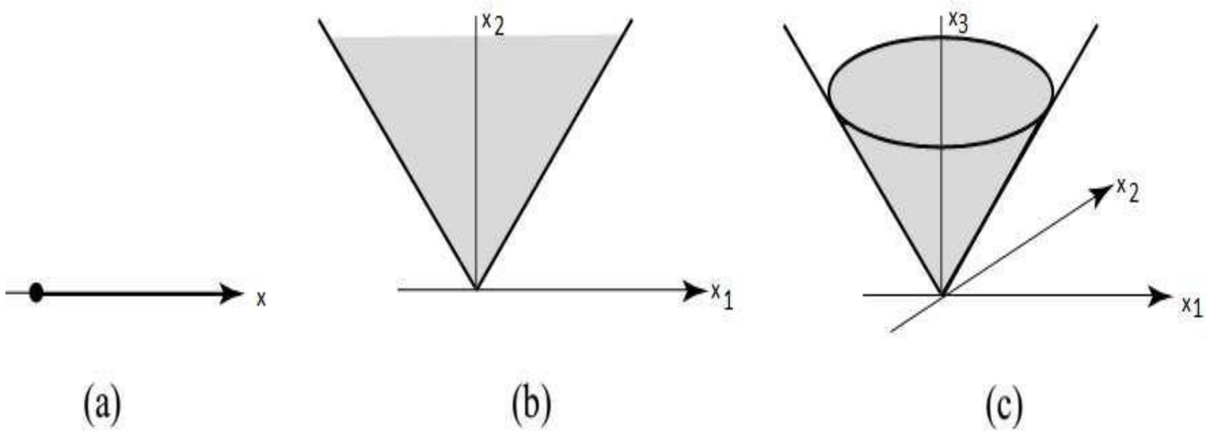

Figure 1 Second-order cones of dimension (a) $n = 1$, (b) $n = 2$, and (c) $n = 3$.

The standard form SOCP can accommodate nonnegative variables. The second-order cone constraints can be used to represent several common constraints. In fact if all cones $K_i$are one-dimensional cone, then Eq. (3.1) is just a standard form of linear programming Eq. (3.4) that is described in section (3.2).

Example (Euclidean) ball with center (x_c) and radius (r),

The dual of the SOCP problem in Eq. (3.1) referred to here after as the dual SOCP problem can be shown to be of the form:

Maximize $b^T y$ (3.2.a)

Subject to: $\widehat{A}_i^{\ T} y + s_i = c_{\hat{i}}$ (3.2b)

$i = 1, \cdots, q$

where $y \in R^{m \times 1}$and $s_i \in R^{n_i \times 1}$. (3.2c)

A SOCP is a convex optimization problem having linear objective function and second- order cone constraints. If we let $x = -y\hat{A}_i^{\ T} = \begin{bmatrix} c_i^{\ T} \\ A_i^{\ T} \end{bmatrix}$and $\widehat{c_i} = \begin{bmatrix} d_i \\ b_i \end{bmatrix}$, then t the SOCP problem in equation (3.2) can be expressed as:

Minimize $b^T x$ (3.3a)

Subject to: $\left\| A_i^{\ T} x + b_i \right\| \leq c_i^{\ T} x + d_i\ ,\ \ i = 1, \cdots, q$ (3.3b)

where $x \in R^n$is the optimization variable, and the problem parameters are$b \in R^n$,$A_i \in R^{(n_i - 1) \times n}$, $A_i \in R^{n_i - 1}$,$c_i \in R^n$ and $d_i \in \mathbb{R}$. The norm appearing in the constraints is the standard Euclidean norm. We call the constraint $\left\| A_i^{\ T} x + b_i \right\| \leq c_i^{\ T} x + d_i$is asecond-order coneconstraint of dimensionn_i, when$n = \sum_{i=1}^{q} n_i$.

The main idea of SOCP is that objective function in optimal problem can be translated into constraint function by introducing variables, then through suitable transform, new objective variable and original variables to be optimized can be combined into new optimized variable, and the original problem constraint can transform into SOCP constraint, finally, as long as objective function and constraint function in the optimization problem can be expressed in the form of SOCP.

SOCPs are representative of a quite large class of convex optimization problems. Indeed, LPs, convex QPs, SOCP and SDP can all be represented as CP problems as illustrated in fig. (3.2) bellow.

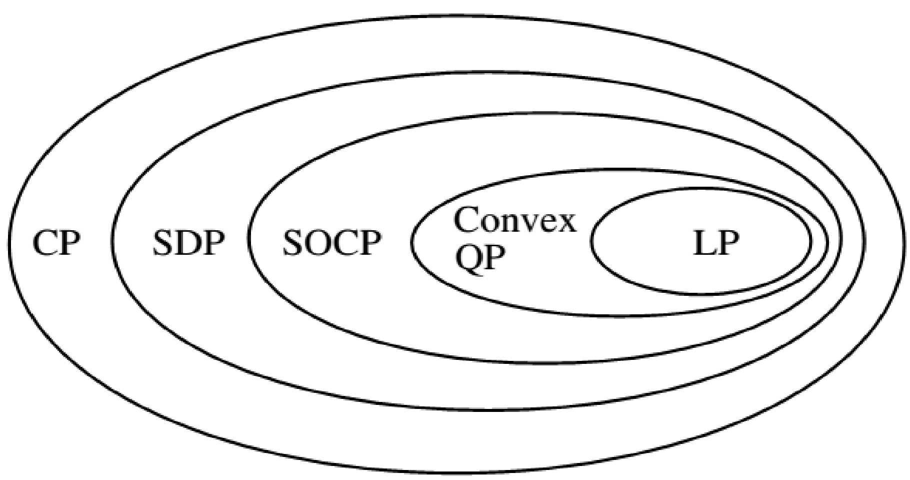

Figure 2 Relations among LP, convex QP, SOCP, SDP, and CP problems.

SOCP problems are a convex optimization problem, which include LP, QP and QCQP problems as special class.

4.1. QUADRATIC PROGRAMMING (QP) PROBLEM

We have already seen that an LP is readily expressed as an SOCP with one dimensional cone. Let us now consider the general convex quadratic programming (QP). QP is a family of methods, techniques, and algorithms that can be used to minimize quadratic objective function subject to linear constraints. On the one hand, QP shares many combinatorial features withlinear programming (LP) and on the other; it is often used as the basis of constrained nonlinear programming. In fact, the computational efficiency of a nonlinear programming algorithm is often heavily dependent on the efficiency of the QP algorithm involved. As a special case, we can formulate a convex quadratic programming (QP) problem

$$\text{Minimize} x^T p x + 2{q_0}^T x + r_0 \quad (3.9a)$$

$$\text{Subject to: } {a_i}^T x \leq b_i,\ i = 1, \dots, q \quad (3.9b)$$

as an SOCP with one constraint of dimension $n+1$ and q constraint of dimension one, where $p$is symmetric $n \times n$positive definite matrices,$q_0$is a vectors in$R^n$and r_0 is scalar. We show that the problem can be converted to second-order cone format. We have:$x^T px + 2{q_0}^T x + r_0 = (p^{\frac{1}{2}}x)^T p^{\frac{1}{2}}x + 2(p^{-\frac{1}{2}}q_0)^T p^{\frac{1}{2}}x + r_0$

$$= (p^{\frac{1}{2}}x)^T p^{\frac{1}{2}}x + 2(p^{-\frac{1}{2}}q_0)^T p^{\frac{1}{2}}x + {q_0}^T p^{-1} q_0 + r_0 - {q_0}^T p^{-1} q_0$$

$$= \left\| (p^{\frac{1}{2}}x) + (p^{-\frac{1}{2}}q_0) \right\|^2 + r_0 - {q_0}^T p^{-1} q_0$$

where p is symmetric and positive semi-definite matrix mean that $p = p^T$and spectral decomposed matrix.

Let $\hat{p} = p^{-1}q_0$be a constant, then minimizing (3.9a) is equivalent to minimizing$\left\| p^{\frac{1}{2}}x + \hat{p} \right\|$and thus the problem in Eq. (3.9) can be formulated as:

$$\text{Minimize} t \quad (3.10a)$$

$$\text{Subject to: } \left\| p^{\frac{1}{2}}x + \hat{p} \right\| \leq t \quad (3.10b)$$

$${a_i}^T x \leq b_{i,} i = 1, \dots, q \quad (3.10c)$$

Where$t$is an upper bound for$\left\| p^{\frac{1}{2}}x + \hat{p} \right\|$that can treated as an auxiliary variable of the

$$\text{Minimize}\, b^T \hat{x} \quad (3.11a)$$

$$\text{Subject to: } \|\bar{e}\hat{x} + \hat{e}\| \le \hat{b}^T \hat{x} \quad (3.11b)$$

$$\hat{A}\hat{x} \le b \quad (3.11c)$$

which is SOCP problem.

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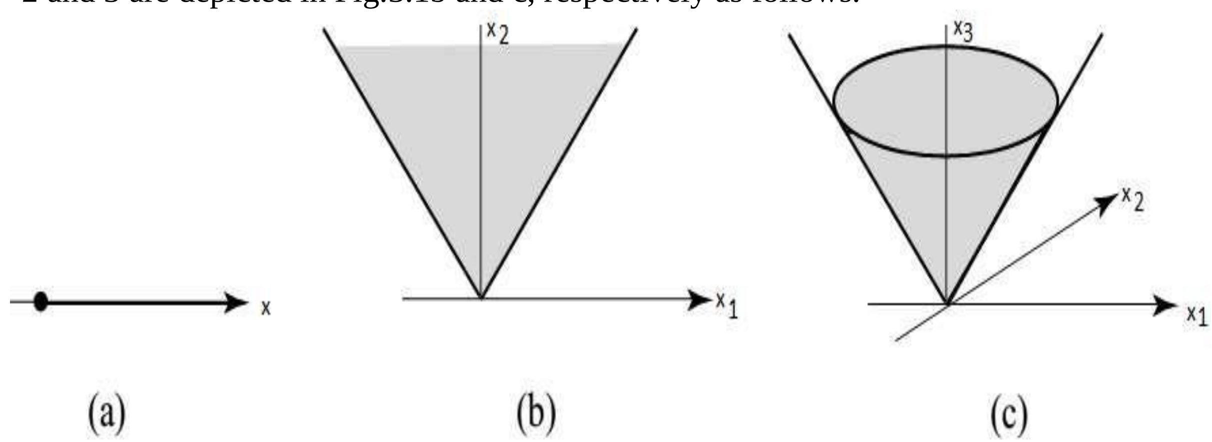
It is interesting to note that the SOCP problems in Eq.(3.1) are involving a linear objective function and a linear equality constraint. Each variable vector $x_i$in an SOCP problem is constrained to the second-order coneK_i. For $n = 1$, the second-order cone degenerates into a ray on the x-axis starting from $x = 0$, as shown in Fig. (3.1a).The second-order cones for $n = 2$ and 3 are depicted in Fig.3.1b and c, respectively as follows.



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For $i = 1, \cdots, q$

where $y \in R^{m\times 1}$ and $s_i \in R^{n_i \times 1}$. (3.2c)

A SOCP is a convex optimization problem having linear objective function and second- order cone constraints. If we let $x = -y\hat{A}_i^{\,T} = \begin{bmatrix} c_i^{\,T} \\ A_i^{\,T} \end{bmatrix}$ and $\hat{c_i} = \begin{bmatrix} d_i \\ b_i \end{bmatrix}$, then t the SOCP problem in equation (3.2) can be expressed as:

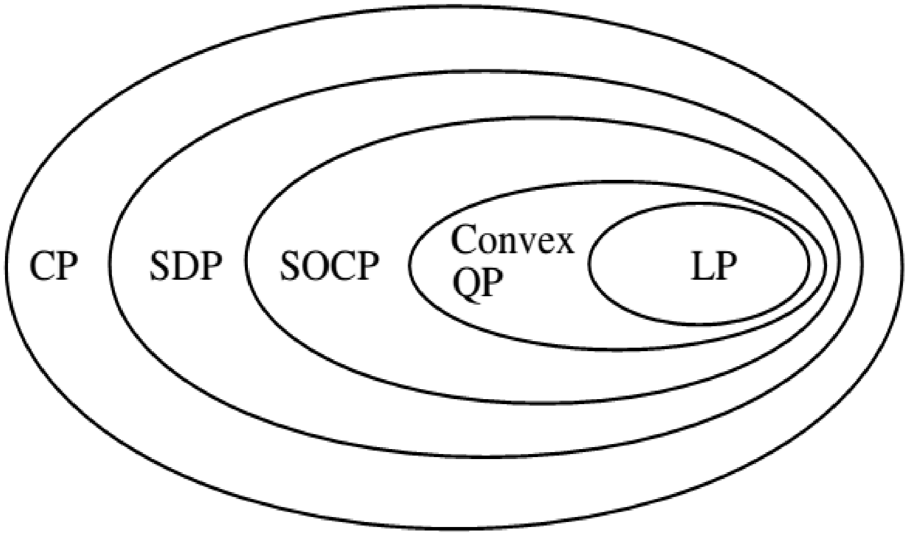
Minimize $b^T x$ (3.3a)

Subject to: $\|A_i^{\,T} x + b_i\| \leq c_i^{\,T} x + d_i$, $i = 1, \cdots, q$ (3.3b)

where $x \in R^n$ is the optimization variable, and the problem parameters are $b \in R^n$, $A_i \in R^{(n_i-1)\times n}$, $A_i \in R^{n_i-1}$, $c_i \in R^n$ and $d_i \in \mathbb{R}$. The norm appearing in the constraints is the standard Euclidean norm. We call the constraint $\|A_i^{\,T} x + b_i\| \leq c_i^{\,T} x + d_i$ is asecond-order coneconstraint of dimension n_i, when $n = \sum_{i=1}^{q} n_i$

The main idea of SOCP is that objective function in optimal problem can be translated into constraint function by introducing variables, then through suitable transform, new objective variable and original variables to be optimized can be combined into new optimized variable, and the original problem constraint can transform into SOCP constraint, finally, as long asobjective function and constraint function in the optimization problem can be expressed in the form of SOCP. SOCPs are representative of a quite large class of convex optimization problems. Indeed, LPs, convex QPs, SOCP and SDP can all be represented as CP problems as illustrated in fig. (3.2) below

Figure 2 Relations among LP, convex QP, SOCP, SDP, and CP problems.



problems as special class.

5.2. QUADRATIC PROGRAMMING (QP) PROBLEM

We have already seen that an LP is readily expressed as an SOCP with one dimensional cone. Let us now consider the general convex quadratic programming (QP). QP is a family of methods, techniques, and algorithms that can be used to minimize quadratic objective function subject to linear constraints. On the one hand, QP shares many combinatorial features withlinear programming (LP) and on the other; it is often used as the basis of constrained nonlinear programming. In fact, the computational efficiency of a nonlinear programming algorithm is often heavily dependent on the efficiency of the QP algorithm involved. As a special case, we can formulate a convex quadratic programming (QP) problem

$$\text{Minimize} x^T p x + 2{q_0}^T x + r_0 \tag{3.4a}$$

$$\text{Subject to: } {a_i}^T x \le b_i,\ i = 1, \dots, q \tag{3.4b}$$

as an SOCP with one constraint of dimension $n+1$ and q constraint of dimension one, where $p$is symmetric $n \times n$positive definite matrices,$q_0$is a vectors in$R^n$and r_0 is scalar. We show that the problem can be converted to second-order cone format. We have:

$$x^T p x + 2{q_0}^T x + r_0 = (p^{\frac{1}{2}}x)^T p^{\frac{1}{2}}x + 2(p^{-\frac{1}{2}}q_0)^T p^{\frac{1}{2}}x + r_0$$

$$= (p^{\frac{1}{2}}x)^T p^{\frac{1}{2}}x + 2(p^{-\frac{1}{2}}q_0)^T p^{\frac{1}{2}}x + {q_0}^T p^{-1} q_0 + r_0 - {q_0}^T p^{-1} q_0$$

$$= \left\| (p^{\frac{1}{2}}x) + (p^{-\frac{1}{2}}q_0) \right\|^2 + r_0 - {q_0}^T p^{-1} q_0$$

where p is symmetric and positive semi-definite matrix mean that $p = p^T$and spectral decomposed matrix.

Let $\hat{p} = p^{-1}q_0$be a constant, then minimizing (3.9a) is equivalent to minimizing$\left\| p^{\frac{1}{2}}x + \hat{p} \right\|$and thus the problem in Eq. (3.9) can be formulated as:

Minimize$t$ (3.5a)

Subject to: $\left\| p^{\frac{1}{2}}x + \hat{p} \right\| \le t$ (3.5b)

${a_i}^T x \le b_{i,} i = 1, \dots, q$ (3.5c)

Where$t$is an upper bound for$\left\| p^{\frac{1}{2}}x + \hat{p} \right\|$that can treat as an auxiliary variable

Minimize$b^T \hat{x}$ (3.6a)

Subject to: $\|\bar{e}\hat{x} + \hat{e}\| \le \hat{b}^T \hat{x}$ (3.6b)

$\hat{A}\hat{x} \le b$ (3.6c)

which is SOCP problem.

5.3. QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING (QCQP)

Quadratically constrained quadratic programming (QCQP) problem is a problem in which both objective function and constraint functions are quadratic. We have already seen that QP problem is readily expressed as an SOCP. Let us now consider the general convex quadratically constrained quadraticprogramming

Minimize $f(x) = x^T p_0 x + 2{q_0}^T x + r_0$ (3.7a)

Subject to:$x^T p_i x + 2{q_i}^T x + r_i \le 0, i = 1, \dots, q$ (3.7b)

where $p_0, p_1, \dots, p_n \in R^n$are symmetric, positive semi-definite and spectral decomposed matrix. We will assume, for simplicity, that the matrices p_i are positive definite, although the problem can be reduced to SOCP in general.

Any convex quadratic constraint of an optimization problem can be rewritten using second-order cone membership constraints. When we have access to a reliable solver for second-order cone optimization, it may be desirable to convert convex quadratic constraints to second ordercone constraints. Fortunately, a simple recipe is available for these conversions. Consider the following quadratic constraint:

$$x^T p_i x + 2{q_i}^T x + r_i \leq 0,$$

This is a convex constraint if the function on the left-hand side is convex which is true if and only if p_i is a positive semi-definite matrix. Let us assume p_i is positive definite for simplicity. Then problem in Eq. (3.12) reformulate similarly as QP

$$x^T p_0 x + 2{q_0}^T x + r_0 = \left\| {p_0}^{\frac{1}{2}} x + {p_0}^{-\frac{1}{2}} q_0 \right\|^2 + r_0 - {q_0}^T {p_0}^{-1} q_0$$

$$x^T p_i x + 2{q_i}^T x + r_i = \left\| {p_i}^{\frac{1}{2}} x + {p_i}^{-\frac{1}{2}} q_i \right\|^2 + r_i - {q_i}^T {p_i}^{-1} q_i \leq 0, i = 1, \dots, q$$

This can be solved via the SOCP with $p+1$ constraints of dimension $n+1$, let t is the upper bound of $\left\| {p_0}^{\frac{1}{2}} x + {p_0}^{-\frac{1}{2}} q_0 \right\|$ then the general convex quadratically constrained quadratic programming problem (3.12) is reformulated as:

Minimize t (3.8a)

Subject to: $\left\| {p_0}^{\frac{1}{2}} x + {p_0}^{-\frac{1}{2}} q_0 \right\| \leq t$ (3.8b) $\left\| {p_i}^{\frac{1}{2}} x + {p_i}^{-\frac{1}{2}} q_i \right\| \leq ({q_i}^T {p_i}^{-1} q_i - r_i)^2$ (3.8c)

Where $t \in R$ is a new optimization variable

The optimal values of problem (3.12) and (3.13) are equal up to a constant and square root. More precisely, the optimal value of the QCQP (3.13) is equal to $p^{*2} + r_0 - {q_0}^T {p_0}^{-1} q_0$ where p^* is the optimal value of SOCP (3.12).

5.4. THE DUAL SOCP

In this section we outline the duality theory for SOCP problems the dual of the SOCP (3.3) is given by:

Maximize $-\sum_{i=1}^{q} ({b_i}^T y_i + d_i s_i)$(3.9)

Subject to: $\sum_{i=1}^{q} (A_i y_i + c_i s_i) = c$

$$\|y_i\| \leq s_i, i = 1, \dots, q$$

where $y_i \in R^{n_i - 1}$ and $s_i \in R^q$ are the dual optimization variables. We denote a set of y_i's, $i = 1, \dots, q$, by y. The dual SOCP (3.14) is also a convex programming problem

We will refer to the SOCP (3.3) as the primal SOCP when we need to distinguish it from the dual. The primal SOCP (3.3) is called feasible if there exists a primal feasible set x, i.e., an x that satisfies all constraints in Eq. (3.3). It is called strictly feasible if there exists a strictly primal feasible x, i.e., an x that satisfies the constraints with strict inequality. The vectors y and $s$are dual feasible if they satisfy the constraints in Eq. (3.14) and strictly dual feasible if in addition they satisfy $\|y_i\| \leq s_i, i = 1,..,q$. We say the dual SOCP (3.14) is strictly feasible if there exist strictly feasible$y_i$and s_i.

The difference between the primal and dual objectives is called the duality gap. The duality gap associated with x, y and s will be denoted by $\eta(x, y, s)$, or simply η.\

$\eta(x, y, s) = c^T x - (-\sum_{i=1}^{q}(b_i{}^T y_i + d_i s_i))$
(3.10)

$$= C^T x + \sum_{i=1}^{q}(b_i{}^T y_i + d_i s_i)$$

The basic facts about the dual problem are:

5.4.1. (Weak duality): the dual objective is less or equal to the primal objective at optimum $p^* \geq d^*$

5.4.2. Strong duality)if the primal and dual problems are strictly feasible, then $p^* = d^*$

5.4.3. If the primal and dual problems are strictly feasible, then there exist primal and dual feasible points that attain the optimal values.

Weak duality corresponds to the fact that the duality gap is always nonnegative, for any feasible x, y and s. To see this, we observe that the duality gap is associated with primal and dual feasible points x, y and s can be expressed as a sum of nonnegative terms, by writing it in the form:

$$\eta(x, y, s) = \sum_{i=1}^{q}(y_i{}^T\left(A_i{}^T x + b_i\right) + s_i(c_i{}^T x + d_i)) = \sum_{i=1}^{q}(y_i{}^T u_i + s_i t_i) \qquad (3.11)$$

Each term in the right- hand sum is nonnegative, where $u_i = A_i{}^T x + b_i$ and $t_i = c_i{}^T x + d_i$.

$$y_i{}^T u_i + s_i t_i \geq \|y_i\|\|s_i\| + s_i t_i \geq 0$$

The first inequality follows from the Cauchy-Schwarz inequality. The second inequality follows from the fact that $t_i \geq \|t_i\| \geq 0$ and $s_i \geq \|y_i\| \geq 0$. Therefore $\eta(x, y, s) \geq 0$ for any feasible $x, y, s$and as immediate consequence we have$y^* \geq d^*$, i.e., weak duality.

5.5. LOGARITHMIC BARRIER FUNCTIONS FOR LP AND SOCP

We now define the notation of barrier function for convex in general and for the LP and SOCP, in particular. Then explain how we can use the barrier functions to find the optimal solutions of these problems. We use the barrier function to convert the constrained optimization to essentially, an unconstrained optimization problem. When we are solving these problems we assume that the constraint set is a convex set and has nonempty interior. When we looked at the Lagrangian methods for minimizing twice differentiable convex function with equality constraints, one of the limitations of this method is that we cannot deal with inequality constraints. The barrier method is away to address this issue.

Definition A barrier function is a continuous function whose values on a point increase to infinity as the point approaches the boundary of the feasible region of an optimization problem.

Let $K \in R^n$ be a convex set with nonempty interior. Then the function g: interior of $K \rightarrow R$ is called a barrier function if it has the following properties

(i) g is convex

(ii) For each sequence of point $x_n$is interior of K such that $\lim_{n\to\infty} x_n$ exists and belongs to interior of K, $\lim_{n\to\infty} g(x_n) = \infty$.

Note that since the domain of $g(x)$is interior of K and by the properties of $g(x)$, the minimum value of g is attained in the interior of K. Now let us explain how we use the barrier functions. Consider the following constrained minimization problem

$$\text{(P)} \quad \text{Minimize } c^T x \qquad (3.12)$$

$$\text{Subject to: } x \in K$$

Let $g(x)$be a barrier function for problem (P).Forgiven, $\mu > 0, \mu \in \mathbb{R}$,by multiplying the barrier function with μ and adding it to the objective functions. We can convert the constrainedminimizationproblemtounconstrainedminimization problem (P_μ):

$$(P_\mu) \text{Minimize } c^T x + \mu \sum_{i=1}^{q} g(x) \qquad (3.13)$$

Let for $\mu > 0, x_\mu{}^*$ be the point at which the minimum of Eq.(3.13)is attained. Since $g(x) \rightarrow \infty$ when x approaches the boundary of K and $P_\mu$is a minimization problem, $x_\mu{}^*$is interior of (K). Suppose we solve this problem for a decreasing sequence ofμ_n. One can show that as $\mu_n. \rightarrow 0$, $x_\mu{}^* \rightarrow x^*$,the optimum of (P). In other words, sequence of optimal points of the problem (P_μ) converges to the optimal point of the original problem (P). In fact, as μ varies towards zero, the set of "μ-optimal" points $x_\mu{}^*$ traverse a smooth path in the interior of K called the central path associated with the barrier function g; it can be shown that this path ends at x^*.

One may think that at the outset by choosing μ very small, we can solve the problem (P_μ) once and get a sufficiently close approximation to x^*.However, it turns out that the problem becomes numerically very ill-conditioned when μ is small. Also, for such a small ▯ it is hard to find a suitable initial solution.Instead a better approach is to first choose μ fairly large; the larger the μ, the easier it is to find an initial solution x_o. Next, one uses one or a few iterations of Newton's method at the current value of μ to get a new point x_1. Then μ is reduced by a constant factor: ▯ ← a▯ for some constant a, and the last result of Newton's method, $x_1$is used as a starting point for the new optimization problem. This process is repeated until $x_1$is sufficiently close to x^*. With judicious choices of the barrier function g$(x), a$, and the initial point x_0, one can show that the procedure just outlined can result in a well-behaved algorithm.

Assuming it is possible to find a strictly feasible point x^*, that is a point satisfying Eq. (3.17), a natural strategy for solving Eq. (3.18) is to decrease the objective function as much as possible while ensuring that the boundary of the feasible set is never crossed. One way to prevent an optimization algorithm from crossing the boundary is to assign a penalty to approaching it. The popularly way of doing this is to augment the objective function by a logarithmic barrier function. At the boundary and therefore presents an optimization algorithm with a barrier to crossing the boundary. Of course, the solution to an inequality constrained is likely to lie on the boundary of the feasible set, so the barrier must be gradually removed by reducing μ to ward zero. Now we find barriers for LP and SOCP problems.

5.6. Logarithmic barrier for LP

Consider the standard form of LP in Eq. (3.4), in this problem, as all the problems to follow; we won't worry about the linear equality constraints in Eq. (3.1b), and focus on the inequality constraints $x_i \geq 0$. The boundary of the nonnegative or than t consists of those where one x_i is zero. By definition any barrier function for the LP problem must approach to ∞ as one of the components of $x = (x_1, x_2, \ldots, x_q)^T$goes to zero. Two examples of barrier functions for the nonnegative or than t are:

$$\sum_{i=1}^{q} \frac{1}{x_i} \text{ and } -\sum_{i=1}^{q} \log x_i \qquad (3.14)$$

Notice that both of this function are convex and approach infinity as any ofx_i approaches zero. Let us explore what we get by applying this barrier to the LP problem:

(P_μ) Minimize $c^T x - \mu \sum_{i=1}^{q} \log x_i$ (3.15)

Subject to: $Ax = b$

where $g(x) = -\sum_{i=1}^{q} \log x_i$is a barrier function and μ is a positive number called the barrier Parameter: The optimal solution to this problem can be found by using Lagrange's theorem: The Lagrangian function for Eq. (3.15) is given by:

$$L(x, y) = c^T x - \mu \sum_{i=1}^{q} \log x_i + \log x_i + y^T (b - Ax) \quad (3.16)$$

where $y \in R^n$ are called Lagrange multipliers. By Lagrange's theorem, $x_\mu$is optimal for Eq. (3.20) if, and only if, the derivatives of Eq. (3.16) with respect to both x and y are zero. That is, we need to solve:

$$\nabla_x L_\mu = c^T - \mu \left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_q} \right) - y^T A = 0 \quad (3.17)$$

$$\Longrightarrow \nabla_Y L_\mu = b - Ax = 0$$

So, we have converted the LP into a nonlinear system of equations. Here we have $m + n$ variablesand $m + n$equations. Since the numbers of variables are equal to the number of equations and by the assumption that the constraint qualification is met, in principle, we can solve this system of equations. If we let:

$$s^T = \mu \left(\frac{1}{s_1}, \frac{1}{s_2}, \dots, \frac{1}{s_q} \right)^T$$

Then by rearranging the equations we get the KKT condition:

$$Ax = b,$$

$$A^T y + s = c, \quad (3.18)$$

$$s_i - \mu x_i^{-1} = 0, \; i = 1, 2, \dots, q.$$

with$X$= diag $(x_1, \dots, x_n)$, $S = s_1, \dots s_n$, and if we multiply the last equation to the left by X, we can rewrite KKT condition as:

$$A^T y + s = c,$$

$$Ax = b, \quad (3.19)$$

$$XS = \mu e, X, S > 0$$

Where e = (1, 1, …, 1)^T we call equation (3.24) the KKT condition for primal and dual problems, due to the convexity of the functions on the feasible set. These conditions are necessary and sufficient for a fixed μ, for a variety of the value of ? as ? approaches to zero.

Notethatthefirsttwosetsofequationsareprimalanddualfeasibilityconditions,for the LP problem, its dual, respectively. In addition the third set of equations, $XS = ?e$ can be interpreted as a relaxation of the complementary slackness condition; thus as μ approaches to zero the complementary slackness conditions will be satisfied by the (x_μ, y_μ, s_μ).

5.7. Logarithmic barrier for SOCP

Now let us define a suitable barrier function for SOCP and find the optimality by using first order necessary condition.

Let us consider the SOCP problems in Eq. (3.3), the corresponding logarithmic barrier function for problems (3.3) is:

$$g(x) = -\sum_{i=1}^{q} \log((c_i{}^T x + d_i)^2 - (\|A_i{}^T x + b_i\|^2) \quad (3.20)$$

with domain $g = \{x: \|A_i{}^T x + b_i\| < c_i{}^T x + d_i, \, i = 1, \dots, q\}$.

$$\nabla g = -2\sum_{i=1}^{q} \frac{1}{(c_i{}^T x + d_i)^2 - (\|A_i{}^T x + b_i\|)^2}((c_i{}^T x + d_i)c_i{}^T - (\|A_i{}^T x + b_i\|)\, A_i{}^T = 0 \quad (3.21)$$

By a similar procedure as LP, the SOCP problem is replaced by:

$$\text{Minimize } c^T x - \mu \sum_{i=1}^{q} \log((c_i{}^T x + d_i)^2 - (\|A_i{}^T x + b_i\|^2) \quad (3.22)$$

where μ is a barrier parameter. The optimal solution to this problem can be found by using first order necessary condition.

$$\nabla_x L = c^T - \frac{2\mu}{(c_i{}^T x + d_i)^2 - (\|A_i{}^T x + b_i\|)^2}(((c_i{}^T x + d_i)c_i{}^T - (\|A_i{}^T x + b_i\|)A_i{}^T = 0 \quad (3.23)$$

by rearranging Eq. (3.23), we can write x in terms of μ, to find optimal solution as μapproaches to zero.

5.8. Illustrative examples

In this section we will see some examples of SOCP problems.

Example 3.1

$$\text{Minimize } x_1 + x_2 \quad (3.24)$$

$$\text{Subject to:} x_1 + 2x_2 = 2$$

$$x_1, x_2 \geq 0$$

Applying the barrier function $g(x) = \sum_{i=1}^{2} \log x_i$, to the objective function we have:

Minimize $x_1 + x_2 - \mu(\log x_1 + \log x_2)$

Subject to: $x_1 + 2x_2 = 2$

Where μ is barrier parameter.

The lagrangian function for Eq. (3.29) is given by:

$$L(x, y) = x_1 + x_2 - \mu(\log x_1 + \log x_2) - y(x_1 + 2x_2 - 2) \quad (3.25)$$

where y is lagrange multiplier. Then x_μ is optimal if and only if the derivatives of Eq.(3.30) with respect to both x and y are zero. That is,

$$\nabla_{x_1} L = 1 - \frac{\mu}{x_1} - y = 0 \quad (3.26a)$$

$$\nabla_{x_2} L = 1 - \frac{\mu}{x_2} - y = 0 \quad (3.26b)$$

$$\nabla_y L = x_1 + 2x_2 = 2 \quad (3.26c)$$

Eq. (3.26a) and (3.26b) we have a relation between x_1 and x_2 such that :

$$\frac{1}{2} + \frac{\mu}{2x_2} = \frac{\mu}{x_1}$$

Then substituting in Eq. (3.26c), we rewrite x_1 and x_2 in terms of μ as:

$$x_1 = \frac{(4\mu + 2) \pm \sqrt{(4\mu + 2)^2 - 16\mu}}{2}$$

$$x_2 = \frac{-(4\mu - 2) \pm \sqrt{(4\mu - 2)^2 + 16\mu}}{4}$$

As μ approaches zero the primal optimal solution is $(x_{1\mu}, x_{2\mu})=(0,1)$ with optimal values $P^* = (x_{1\mu}, x_{2\mu})=1$

If we let $s_1 = \frac{\mu}{x_1}$ and $s_2 = \frac{\mu}{x_1}$ in Eq (3.31), then by rearranging the equation, we get the KKT condition:

$$x_1 + 2x_2 = 2 \quad (3.27a)$$

$$y + s_1 = 1 \quad (3.27b)$$

$$2y + s_2 = 1 \quad (3.27c)$$

$$x_1 s_1 = ▯ \quad (3.27d)$$

$$x_2 s_2 = ▯ \quad (3.27e)$$

Note that the Eq. (3.32a) and (3.32b and 3.32c) are primal and dual feasibility condition for LP problem Eq. (3.29) and it's dual, respectively. In addition, the Eq. (3.27d and 3.27e) can be interpreted as the complementary slackness conditions; thus as μ approach's to zero, we get: $(x_{1\mu}, x_{2\mu})$ = (0,1) and $(s_{1\mu}, s_{2\mu}, y_{\mu})$ = ($^1/_2$, 0, $^1/_2$) are primal and dual feasible solutions, respectively. Duality gap is zero.

6. SUMMARY

SOCPs are non-linear convex optimizations that include linear program (LP), quadratic program (QP) and quadratically constrained quadratic programs (QCQP) can all be formulated as SOCP problems.

When the second order cone constraints have the standard Euclidean norm form, we have a SOCP in standard inequality form:

$$\text{Minimize } c^T x$$

$$\text{Subject to:} \left\| A_i{}^T x + c_i \right\| \leq b_i{}^T x + d_i, i = 1, 2, \dots, q,$$

Where $x \in R^n$ is the optimization variable, and the problem parameters are $c \in R^n$,$A_i \in R^{(n_i{}^{-1})} \times n, b_i \in R^{n_i - 1}, c_i \in R^n$ and $d_i \in \mathbb{R}$.

And its dual is given by:

Maximize -$\sum_{i=1}^{q} b^T y_i + d_i s_i$

Subject to $\sum_{i=1}^{q} A_i y_i + c_i s_i = f$

where $\|y_i\| \leq s_i, i = 1, \dots., q$

When$n_i = 1$, the SOCP reduces to the linear program. Linear programming and second order cone programming are both linear objective and constraint functions.

The QP and QCQP problems

Minimize $x^T e x + 2 q_0{}^T x + r_0$

Subject to:$a_i{}^T x \geq b_i, i = 1, \dots, q$

and

Minimize f(x) = $x^T e_0 x + 2 q^T x + r_0$

Subject to:$x^T e x + 2 q_i{}^T x + r_i \leq 0, i = 1, \dots, q$ are reformulated as:

Minimize t

Subject to:$\left\| e^{\frac{1}{2}} x + e \right\| \leq t,$

$$a_i{}^T x \geq b_i,\ i = 1, \dots, q,$$

LP and SOCP problems can be solved by logarithmic barrier method, first by finding the barrier function and then adding to the objective function as

$$\text{Minimize} c^T x - \mu \log(c_i{}^T x + d_i)^2 - (\|A_i{}^T x + b_i\|)^2$$

Therefore, by applying first order necessary condition, we can approximate the optimal solution of theproblem.

7. REFERENCES

Andersen, E.D., Roos C. and Terlaky, T. 2000. *On Implementing A primal-dual interior-point.* David G. luenberger and Yinyu Ye. 2008. *Linear and nonlinear programming*, *3rd edition*. Springer-Science + Business Media, LLC, Stanford University.

Antoniou A. and Wu-Sheng, Lu. 2007. *Practical Optimization: Algorithms and Engineering* Applications Springer Science, Business Media, LLC. PP, 373-518.

Ashok D., Belegundu and Tirupathi R. Chandrupatla. 2011. *Optimization Concepts and Applications in Engineering. 2ndedition*. Cambridge University press, United States of America.

Dantzig B., Mukund and Thapa T. 2003.*Linear programming: Theory and Extension*. Springer- Science and Business Media, New York.

David G., Luenberger and Yinyu Ye. 2008. *Linear and nonlinear programming*. Third edition. Frederisk S. hillier, series Editor, Stanford University.

Edwin k. P., Chong and Stanislaw H. Zak. 2001. *An Introduction to Optimization 2nd Edition*. by John Wiley & Sons, Inc. Published simultaneously in Canada

FaridAlizadeh. 2001. *Semidefinite and second order cone programming seminar lecture notes.* New York, NY,USA

Igor Griva, Stephen G. Nash, ArielaSofer, 2009. *Linear and nonlinear optimization*, *2ndedition*. by the Society for Industrial and Applied Mathematics, George Mason University Fairfax, Virginia

JacekGondzio. 2016 *Interior point methods: Semidefinite programming and SOCP.* URL:htt;/www.maths.ed.ac.uk/gondzio

Magnus A. 2004. *A Second-Order Cone Programming Algorithm For Model Predictive Control. Royal Institute of Technology, Sweden.*

Miguel, S., Lieven V., Stephen B. and Hervel L. 1998.*Application of Second Order Cone Programming*.Information Systems Laboratory, Electrical Engineering Department, Stanford University.

Robert J.V. and Hande Y. 1991.*Using LOQO to Solve Second Order Cone Programming Problems.*Statistics and Operation Research. Princeton, University

Robert R. 2012. *Interior-Point Methods and Linear Programming*.University of Toronto, NewYork.

Singiresu, S. Rao, Wiley J. and Sons. 2009. *Engineering optimization Theory and Practice. Fourth edition.*

Stephen B. and Lieven V. 2004. *Convex Optimization*.United State of America by Cambridge University Press, NewYork.

Thomas, R.K. 2008.*Interior Point Algorithms for Quadratic Programming*.Technical University of Denmark.

Wenyu S. and Ya-xiang Y. 2006. *Optimization Theory and Methods Non-linear Programming.*Springer Science + Business Media, NewYork.