"A STUDY OF EULERIAN INTEGRALS OF MULTIPLE HYPERGEOMETRIC FUNCTION WITH GENERAL ARGUMENT"

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ABSTRACT

Our aim to present this paper is to investigate the most general nature of the functions and polynomials occurring in the various integrals, we evaluate three unified Eulerian integrals. Our results provide interesting unifications and extensions of a large number of new and known results. Our integrals involve simpler polynomials such as, Jacobi polynomials, Gould & Hopper polynomials, Laguerre polynomials of several variables.

INTRODUCTION

In this research paper, we evaluate three unified Eulerian integrals. The first of these integrals involve the product of a general class polynomials $S_V^U[x]$, the generalized sequence of function $S_n^{\alpha,\beta,O}[x]$ and the multivariable Hypergeometric function with general argument which is giving by the multiple series by Srivastava and Daoust (1969). The second integral involve the product of a general class of multivariable polynomials $S_{V_1,\ldots\ldots,V_r}^{U_1,\ldots\ldots,U_r}[x_1,\ldots\ldots\ldots.,x_r]$ and the multivariable Hypergeometric function where as the last integral contain the product of another class of multivariable polynomials $S_V^{U_1,\ldots\ldots,U_r}[x_1,\ldots\ldots\ldots,x_r]$ which was introduce by Srivastava and Garg (1987) and the multivariable Hypergeometric function in their integrals.

1.1 Theorem:1 Eulerian First Integral:

Prove that

$$\int_0^1 x^{m(a-1)}(1-x^m)^{b-1} S_V^U\left[yx^{m\mu}(1-x^m)^v\right] S_n^{\alpha,\beta,0}\left[hx^{m\xi}(1-x^m)^{\xi}\right]$$

$$\times F^{p:q_1;\ldots\ldots;q_n}_{1;m_1;\ldots\ldots;m_n}\left(\begin{matrix}(a_p):(b_{q_1});\ldots\ldots\ \ldots\ldots;(b_{q_n}^{(n)});\\(\alpha_1):(\beta_{m_1});\ldots\ldots\ldots\ ;(\beta_{m_n}^{(n)});\end{matrix}\ z_1x^{\lambda},\ldots,z_nx^{\lambda}\right)dx$$

$$= m\frac{\Gamma(\alpha+\frac{1}{m}-1)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+\frac{1}{m}-1)}\sum_{k=0}^{(\frac{V}{U})}\phi_1(k,v,u,e,p)\ F^{p+\lambda:q_1;\ldots\ldots;q_n}_{1+\lambda;m_1;\ldots\ldots;m_n}$$

$$\times\left(\begin{matrix}(a_p):\frac{(\alpha+\frac{1}{m}-1)}{\lambda},\ldots\ldots\ \ldots,\frac{(\alpha+\lambda+\frac{1}{m}-2)}{\lambda};\ (b_{q_1}^{1});\ldots\ldots;(b_{q_n}^{(n)});\\(\alpha_1):\frac{(\alpha+\beta+\frac{1}{m}-1)}{\lambda},\ldots\ldots,\frac{(\alpha+\beta+\lambda+\frac{1}{m}-2)}{\lambda};\ (\beta_{m_1}^{1});\ldots;(\beta_{m_n}^{(n)});\end{matrix}\ z_1,\ldots,z_n\right) \tag{1.1}$$

Then the p.d.f form of (1.1) is as below

$$F(x)=\frac{x^{m(a-1)}(1-x^m)^{b-1}S_V^U[yx^{mu}(1-x^m)^v]S_n^{\alpha,\beta,0}[hx^{m\xi}(1-x^m)^{\xi}]F^{p:q_1;\ldots\ldots;q_n}_{1;m_1;\ldots\ldots;m_n}[\chi_1]}{m\frac{\Gamma(\alpha+\frac{1}{m}-1)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+\frac{1}{m}-1)}\sum_{k=0}^{(\frac{V}{U})}\phi_1(k,v,u,e,p)F^{p+\lambda:q_1;\ldots\ldots;q_n}_{1+\lambda;m_1;\ldots\ldots;m_n}[\chi_2]}$$

= 0 elsewhere (1.2)

Where

$$\chi_1 = \begin{bmatrix} (a_p):(b_{q_1});\ldots\ldots\ \ldots;(b_{q_n}^{(n)}); \\ (\alpha_1):(\beta_{m_1});\ldots\ldots\ ;(\beta_{m_n}^{(n)}); \end{bmatrix} z_1 x^{\lambda},\ldots, z_n x^{\lambda}$$ and

$$\chi_2 = \left(\begin{matrix} (a_p): \dfrac{\alpha + \frac{1}{m} - 1}{\lambda}, \ldots\ldots, \dfrac{(\alpha + \lambda + \frac{1}{m} - 2)}{\lambda} : (b_{q_1});\ldots\ldots\ \ldots;(b_{q_n}^{(n)}); \\ (\alpha_1): \dfrac{(\alpha + \beta + \frac{1}{m} - 1)}{\lambda}, \ldots\ldots, \dfrac{(\alpha + \beta + \lambda + \frac{1}{m} - 2)}{\lambda}; (\beta_{m_1});\ldots\ldots; (\beta_{m_n}^{(n)}); \end{matrix} \; z_1, \ldots, z_n \right)$$

Where $S_V^U[x]$ be the general class of polynomials, $S_n^{\alpha,\beta,0}[x]$ be the generalized sequence of function. These are very general in nature and extends a number of classical polynomials introduced and studied by various research work such as Gould and Hopper (1962), Krall and Frink (1949), Singh(1981), Singh and Srivastava (1964), Dhilion(1989), etc and the multivariable Hypergeometric function which we occurring in(1.1). Here we take x^m instead of x in the above definitions and we find such new results.

Where

$$\phi_1(k, v, u, e, p) = \frac{(-V)_{UK}\, A(V, K)\, \theta_1(v, u, e, p) h^R}{K!} y^k \quad (1.3)$$

$$\alpha = a + \mu k + \xi R \;\&\; \beta = b + \vartheta + \zeta R$$

(1.4)

Also the following conditions are assumed to be satisfied.

(i) Re (a, b, r) › 0, Re (α, β) › 0

(ii) Min (μ, v, θ, λ, ξ, ζ, p, q_1) $\geq$ 0 (j = 1, 2, ...,n) (not all zero simultaneously)

(iii) (a) $\mu \geq 0$, $v \geq 0$, $\Delta_1 > 0$, $\Delta_2 > 0$

(b) $\mu < 0$, $v < 0$, $\Delta_1 + \mu\,[V/U] > 0$, $\Delta_2 + v\,[V/U] > 0$

(c) $\mu \geq 0$, $v < 0$, $\Delta_1 > 0$, $\Delta_2 + v\,[V/U] > 0$

(d) $\mu < 0$, $v \geq 0$, $\Delta_1 + \mu\,[V/U] > 0$, $\Delta_2 > 0$

Where

$$\Delta_1 = \mathrm{Re}(a) + \xi\,\{(1(m+n) + p + \sum_{j=1}^{n} q_j$$

$$\Delta_2 = \mathrm{Re}(a) + \xi\,\{(1(m+n) + p + \sum_{j=1}^{n} q_j$$

Proof:- To prove the first integral (1.1), We first express the general class of polynomials S_V^U [x] and the generalized sequence of function $S_n^{\alpha,\beta,0}$ [x] occurring in its left hand side in their respective series forms.

The generalized sequence of function is

$$S_n^{\alpha,\beta,0}[x] = \sum_{v,\,u,\,e,\,p} \theta_1\,(v, u, e, p)\,x^R \quad \ldots\ldots \quad (1.5)$$

The general class of polynomials is introduced as

$$S_V^U[x] = \sum_{K=0}^{(\frac{V}{U})} \frac{(-V)_{UK}\,A(V,\ K)}{K!}\,x^K \qquad N = 0,1,2\ldots.. \quad \ldots\ldots.. \quad (1.6)$$

Here we use the above formula by taking x^m instead of x. While (1.6) is certainly useful generalization, it seems that for some purposes, multiple Hypergeometric function which are of a less general nature are of more immediate value. To this end, we consider the generalized Kamp' ede Fe'riet function first given by Karlson (1973) which we define later.

$$\int_0^1 x^{m(a-1)}(1-x^m)^{b-1} S_V^U\left[yx^{m\mu}(1-x^m)^v\right] S_n^{\alpha,\beta,0}\left[hx^{m\xi}(1-x^m)^{\zeta}\right]$$

$$F^{p:q_1;\ldots\ldots;q_n}_{1;m_1;\ldots\ldots;m_n}\left(\begin{matrix}(a_p):(b_{q_1});\ldots\ldots\ \ldots\ldots;(b_{q_n}^{(n)});\\(\alpha_1):(\beta_{m_1});\ldots\ldots\ldots\ ;(\beta_{m_n}^{(n)});\end{matrix}\ z_1x^{\lambda},\ldots,z_nx^{\lambda}\right)dx \quad (1.7)$$

$$\int_0^1 x^{m(a-1)}(1-x^m)^{b-1}\sum_{k=0}^{(\frac{V}{U})}\frac{(-V)_{UK}\,A(V,K)}{K!}y^kx^{m\mu k}(1-x^m)^{vk}\sum_{k=0}^{(\frac{V}{U})}\theta_1(k,v,u,e,p)h^Rx^{m\xi R}$$

$$\times(1-x^m)^{\zeta R}\ F^{p:q_1;\ldots\ldots;q_n}_{1;m_1;\ldots\ldots;m_n}\left(\begin{matrix}(a_p):(b_{q_1});\ldots\ldots\ \ldots\ldots;(b_{q_n}^{(n)});\\(\alpha_1):(\beta_{m_1});\ldots\ldots\ ;(\beta_{m_n}^{(n)});\end{matrix}\ z_1x^{\lambda},\ldots,z_nx^{\lambda}\right)dx \quad (1.8)$$

Now Using the Multivariable Hypergeometric Function which is giving by Srivastava and Daoust (1969) have giving the multiple series.

By using multivariable Hypergeometric function as:

$$F^{p:q_1;\ldots\ldots;q_{(n)}}_{1;m_1;\ldots\ldots;m_{(n)}}\begin{pmatrix}z_1\\ \vdots\\ \vdots\\ z_n\end{pmatrix}\equiv F^{p:q_1;\ldots\ldots;q_{(n)}}_{1;m_1;\ldots\ldots;m_{(n)}}\left(\begin{matrix}(a_p):(b_{q_1});\ldots\ldots\ \ldots\ldots;(b_{q_n}^{(n)});\\(\alpha_1):(\beta_{m_1});\ldots\ldots\ldots\ ;(\beta_{m_n}^{(n)});\end{matrix}\ z_1,\ldots,z_n\right)$$

$$=\sum_{s_1,\ldots\ldots,s_n=0}^{\infty}A(s_1,\ldots\ldots,s_n)\frac{z_1^{s_1}}{s_1!}\ldots\ldots\frac{z_n^{s_n}}{s_n!} \quad (1.9)$$

Where

$$A(s_1,\ldots\ldots,s_n)\ \frac{\prod_{j=1}^{p}(a_j)_{s_1+\ldots\ldots+s_n}\prod_{j=1}^{q_1}(b_j)_{s_1\ldots\ldots}\prod_{j=1}^{q^{(n)}}(b_j^{(n)})_{s_n}}{\prod_{j=1}^{1}(\alpha_j)_{s_1+\ldots\ldots+s_n}\prod_{j=1}^{m_1}(\beta_j)_{s_1\ldots\ldots}\prod_{j=1}^{m_n}(\beta_j^{(n)})_{s_n}}$$

Putting value $F^{p:q_1;\ldots\ldots;q_n}_{1;m_1;\ldots\ldots;m_n}\left(z_1x^{\lambda},\ldots,z_nx^{\lambda}\right)$ with the help of (1.9), the equation (1.8) becomes

$$\int_0^1 x^{m(a-1)}(1-x^m)^{b-1}\sum_{k=0}^{(\frac{V}{U})}\frac{(-V)_{UK}A(V,K)}{K!}y^k x^{m\mu k}(1-x^m)^{vk}\sum_{k=0}^{(\frac{V}{U})}\theta_1(v,u,e,p)h^R x^{m\xi R}$$

$$\times(1-x^m)^{\zeta R}\sum_{m_1,\ldots\ldots,m_1=0}^{\infty}\Omega(m_1,\ldots..,m_n)\frac{z_1^{m_1}x^{\lambda^{m_1}}}{m_1!}\ldots\ldots\frac{z_n^{m_n}x^{\lambda^{m_1}}}{m_1!}dx$$

Where

$$\Omega(m_1,\ldots..,m_n)=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j+\ldots..+m_n\theta_j^{(n)}}\prod_{j=1}^{B}(b_j)_{m_1\phi_j+\ldots..}\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_1\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j+\ldots..+m_n\psi_j^{(n)}}\prod_{j=1}^{D}(d_j)_{m_1\delta_j+\ldots..}\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

$$=\sum_{k=0}^{(\frac{V}{U})}\frac{(-V)_{UK}A(V,K)}{K!}y^k\sum_{k=0}^{(\frac{V}{U})}\theta_1(v,u,e,p)h^R\int_0^1 x^{m(a-1)}(1-x^m)^{b-1}x^{muk}(1-x^m)^{vk}x^{m\xi R}$$

$$\times(1-x^m)^{\zeta R}\times\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j+\ldots..+m_n\theta_j^{(n)}}\prod_{j=1}^{B}(b_j)_{m_1\phi_j+\ldots..}\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_1\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j+\ldots..+m_n\psi_j^{(n)}}\prod_{j=1}^{D}(d_j)_{m_1\delta_j+\ldots..}\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}\frac{z_1^{m_1}x^{\lambda^{m_1}}}{m_1!}\ldots\ldots\frac{z_n^{m_n}x^{\lambda^{m_1}}}{m_1!}dx$$

$$=\sum_{k=0}^{(\frac{V}{U})}\frac{(-V)_{UK}A(V,K)}{K!}y^k\sum_{k=0}^{(\frac{V}{U})}\theta_1(v,u,e,p)h^R\int_0^1 (x^m)^{a+\mu k+\xi R-1}(1-x^m)^{b+vR}{}_{+}{}^{\zeta R-1}$$

$$\times \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j+.....+m_n\theta_j^{(n)}} \prod_{j=1}^{B}(b_j)_{m_1\phi_j+.....} \prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j+.....+m_n\psi_j^{(n)}} \prod_{j=1}^{D}(d_j)_{m_1\delta_j+.....} \prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}} \frac{z_1^{m_1} x^{\lambda m_1}}{m_1!} .... \frac{z_n^{m_n} x^{\lambda m_1}}{m_1!} dx$$

$$= \sum_{k=0}^{(\frac{V}{U})} \frac{(-V)_{UK} A(V, K)}{K!} y^k \sum_{k=0}^{(\frac{V}{U})} \theta_1(v, u, e, p) h^R \int_0^1 x^{m(\alpha-1)} (1-x^m)^{\beta-1}$$

$$\times \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j+.....+m_n\theta_j^{(n)}} \prod_{j=1}^{B}(b_j)_{m_1\phi_j+.....} \prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_1\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j+.....+m_n\psi_j^{(n)}} \prod_{j=1}^{D}(d_j)_{m_1\delta_j+.....} \prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}} \frac{z_1^{m_1} x^{\lambda m_1}}{m_1!} .... \frac{z_n^{m_n} x^{\lambda m_1}}{m_1!} dx$$

(1.10)

Where $\alpha = a + \mu k + \xi R$ and $\beta = b + \vartheta k + \zeta R$

$$= \sum_{k=0}^{(\frac{V}{U})} \frac{(-V)_{UK} A(V, K)}{K!} y^k \sum_{k=0}^{(\frac{V}{U})} \theta_1(v, u, e, p) h^R$$

$$\times \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j+.......... .....+m_n\theta_j^{(n)}} \prod_{j=1}^{B}(b_j)_{m_1\phi_j+.... ....} \prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_1\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j+.......... ....+m_n\psi_j^{(n)}} \prod_{j=1}^{D}(d_j)_{m_1\delta_j+..........} . \prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

$$\times \int_0^1 x^{m(\alpha+\frac{1}{m}-1)+\lambda m_1 .... \lambda m_n - 1} (1-x^m)^{\beta-1} \frac{z_1^{m_1}}{m_1!} .... \frac{z_n^{m_n}}{m_n!} dx \qquad \textbf{(1.11)}$$

Where $\int_0^1 x^{m(\alpha-1)}(1-x^m)^{\beta-1}dx = \int_0^1 x^{(m\alpha-m+1-1)}(1-x^m)^{\beta-1}dx = \int_0^1 x^{m(\alpha+\frac{1}{m}-1)-1}(1-x^m)^{\beta-1}dx$

By using beta integral now applying Euler Integrals

$$\int_0^1 x^{a-1}(1-x)^{b-1}f(x)\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\sum_{n}^{\infty}\frac{(a)_n C_n}{(\alpha+b)_n}$$

Above formula can be transformed into the following as

$$\int_0^1 x^{m(\alpha+\frac{1}{m}-1)-1}(1-x^m)^{\beta-1}dx = \mathrm{m}\frac{\Gamma(\alpha+\frac{1}{\mathrm{m}}-1)\Gamma)\beta)}{\Gamma(\alpha+\beta+\frac{1}{\mathrm{m}}-1)}\sum_{n}^{\infty}\frac{(\alpha+\frac{1}{\mathrm{m}}-1)_n C_n}{(\alpha+\beta+\frac{1}{\mathrm{m}}-1)_n} \tag{1.12}$$

Where $f(x) = \sum_{n}^{\infty} C_n x^n$, putting the formula (1.12) in (1.11), we get

$$= \sum_{k=0}^{(\frac{V}{U})} \frac{(-V)_{UK}\,A(V,\ K)}{K!} y^k \sum_{k=0}^{(\frac{V}{U})} \theta_1\,(v, u, e, p)h^R$$

$$\times \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j+\dots+m_n\theta_j^{(n)}}\prod_{j=1}^{B}(b_j)_{m_1\phi_j+\dots}\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j+\dots+m_n\psi_j^{(n)}}\prod_{j=1}^{D}(d_j)_{m_1\delta_j+\dots}\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

$$\times\ \mathrm{m}\frac{\Gamma(\alpha+\frac{1}{\mathrm{m}}-1)\ \Gamma(\beta)}{\Gamma(\alpha+\beta+\frac{1}{\mathrm{m}}-1)}\ \frac{(\alpha+\frac{1}{\mathrm{m}}-1)_{\lambda m_1+\dots\dots+\lambda m_n}(\beta)_{\lambda m_1+\dots\dots+\lambda m_n}}{(\alpha+\beta+\frac{1}{\mathrm{m}}-1)_{\lambda m_1+\dots\dots+\lambda m_n}(\alpha+\beta+\frac{1}{\mathrm{m}})_{\lambda m_1+\dots\dots+\lambda m_n}}$$

$$= \mathrm{m}\frac{\Gamma(\alpha+\frac{1}{\mathrm{m}}-1)\Gamma)\beta)}{\Gamma(\alpha+\beta+\frac{1}{\mathrm{m}}-1)}\sum_{k=0}^{(\frac{V}{U})}\phi_1\,(k,v,\ u,\ e,\ p)\times$$

$$\times\ F^{p+\lambda:q_1;\ldots\ldots;q_n}_{1+\lambda;\,m_1;\ldots\ldots;m_n}$$

$$\left(\begin{matrix}(a_p):\dfrac{\alpha+\frac{1}{m}-1}{\lambda},\ldots\ldots,\dfrac{(\alpha+\frac{1}{m}-1)+\lambda-1}{\lambda}: & (b_{q_1});\ldots\ldots\ \ldots;(b_{q_n}^{(n)}); \\ (\alpha_1):\dfrac{(\alpha+\beta+\frac{1}{m}-1)}{\lambda},\ldots\ldots,\dfrac{(\alpha+\beta+\frac{1}{m}-1)+\lambda-1}{\lambda} & (\beta_{m_1});\ldots\ldots;(\beta_{m_n}^{(n)});\end{matrix}\ z_1,\ldots,z_n\right)$$

Where $\sum_{k=0}^{(\frac{V}{U})}\phi_1(k,v,u,e,p)=\sum_{k=0}^{(\frac{V}{U})}\frac{(-V)_{UK}\,A(V,K)}{K!}y^k\sum_{k=0}^{(\frac{V}{U})}\theta_1(v,u,e,p)h^R$

$$= m\frac{\Gamma(\alpha+\frac{1}{m}-1)\Gamma(\beta)}{\Gamma(\alpha+\beta+\frac{1}{m}-1)}\sum_{k=0}^{(\frac{V}{U})}\phi_1(k,v,u,e,p)\times$$

$$\times\ F^{p+\lambda:q_1;\ldots\ldots;q_n}_{1+\lambda;\,m_1;\ldots\ldots;m_n}$$

$$\left(\begin{matrix}(a_p):\dfrac{\alpha+\frac{1}{m}-1}{\lambda},\ldots\ldots,\dfrac{(\alpha+\lambda+\frac{1}{m}-2)}{\lambda}: & (b_{q_1});\ldots\ldots\ \ldots;(b_{q_n}^{(n)}); \\ (\alpha_1):\dfrac{(\alpha+\beta+\frac{1}{m}-1)}{\lambda},\ldots\ldots,\dfrac{(\alpha+\beta+\lambda+\frac{1}{m}-2)}{\lambda}; & (\beta_{m_1});\ldots\ldots;(\beta_{m_n}^{(n)});\end{matrix}\ z_1,\ldots,z_n\right)$$

(1.13)

Then pdf of (1.13) will becomes as

$$F(x) = \frac{x^{m(a-1)}(1-x^m)^{b-1} S_V^U [yx^{mu}(1-x^m)^v] S_n^{\alpha,\beta,0}[hx^{m\xi}(1-x^m)^{\xi}] F_{1;m_1;\ldots;m_n}^{p:q_1;\ldots;q_n}[\chi_1]}{m \frac{\Gamma(\alpha+\frac{1}{m}-1)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+\frac{1}{m}-1)} \sum_{k=0}^{(\frac{V}{U})} \phi_1(k, v, u, e, p) F_{1+\lambda;m_1;\ldots;m_n}^{p+\lambda:q_1;\ldots;q_n}[\chi_2]}$$

$$= \mathbf{0 \text{ elsewhere}} \qquad (1.14)$$

Where

$$\chi_1 = \left[\begin{matrix}(a_p):(b_{q_1});\ldots\ldots;(b_{q_n}^{(n)});\\(\alpha_1):(\beta_{m_1});\ldots\ldots;(\beta_{m_n}^{(n)});\end{matrix}\; z_1 x^{\lambda},\ldots, z_n x^{\lambda}\right]$$ **and**

$$\chi_2 = \left(\begin{matrix}(a_p):\frac{\alpha+\frac{1}{m}-1}{\lambda},\ldots\ldots,\frac{(\alpha+\lambda+\frac{1}{m}-2)}{\lambda}:(b_{q_1});\ldots\ldots;(b_{q_n}^{(n)});\\(\alpha_1):\frac{(\alpha+\beta+\frac{1}{m}-1)}{\lambda},\ldots\ldots,\frac{(\alpha+\beta+\lambda+\frac{1}{m}-2)}{\lambda};(\beta_{m_1});\ldots\ldots;(\beta_{m_n}^{(n)});\end{matrix}\; z_1,\ldots, z_n\right)$$

Hence First integral is proved.

2.1 Theorem:2 Eulerian second Integral Involving $S_{V_1;\ldots;V_r}^{U_1;\ldots;U_r}$ [$x_1 \ldots x_r$]

Prove that

$$\int_0^1 x^{m(a-1)}(1-x^m)^{b-1} S_{V_1;\ldots;V_r}^{U_1;\ldots;U_r}\left[y_1 x^{m\mu}(1-x^m)^{v_1},\ldots, y_r x^{m\mu_r}(1-x^m)^{v_r}\right]$$

$$F_{1;m_1;\ldots;m_n}^{p:q_1;\ldots;q_n}\left(\begin{matrix}(a_p):(b_{q_1});\ldots\ldots;(b_{q_n}^{(n)});\\(\alpha_1):(\beta_{m_1});\ldots\ldots;(\beta_{m_n}^{(n)});\end{matrix}\; z_1 x^{m\lambda}(1-x^m)^{\lambda},\ldots, z_n x^{m\lambda}(1-x^m)^{\lambda}\right) dx$$

$$= m\frac{\Gamma(\alpha_1+\frac{1}{m}-1)\ \Gamma(\beta_1)}{\Gamma(\alpha_1+\beta_1+\frac{1}{m}-1)}\sum_{k_1=0}^{(\frac{V_1}{U_1})}\cdots\sum_{k_r=0}^{(\frac{V_r}{U_r})}(-v_t)_{U_1K_1}\ldots(-v_r)_{U_rK_r}\,A(V_1,k_1;\ldots V_r,k_r)\prod_{i=1}^{r}\frac{y_i^{k_i}}{k_1!}$$

$$\times\, F^{p+\lambda:q_1;\ldots\ldots q_n}_{1+\lambda;\,m_1;\ldots\ldots m_n}\,\times$$

$$\times\left(\begin{matrix}(a_p):\frac{\alpha_1+\frac{1}{m}-1}{\lambda},\ldots\ldots,\frac{(\alpha_1+\lambda+\frac{1}{m}-2)}{\lambda}: & (b_{q_1});\ldots\ldots;(b_{q_n}^{(n)}); \\ (\alpha_1):\frac{(\alpha_1+\beta_1+\frac{1}{m}-1)}{\lambda},\ldots\ldots,\frac{(\alpha_1+\beta_1+\lambda+\frac{1}{m}-2)}{\lambda} & (\beta_{m_1});\ldots;(\beta_{m_n}^{(n)});\end{matrix}\ z_1,\ldots,z_n\right)$$

(2.1)

Then p.d.f form of (2.1) is giving below:

$$F(x)=\frac{x^{ma-1}(1-x^m)^{b-1}S^{u_1;\ldots\ldots v_r}_{v_1;\ldots\ldots v_r}[y_1x^{mu}(1-x^m)^{v_1}\ldots\, y_rx^{mu_r}(1-x^m)^{v_r}]F^{p:q_1;\ldots\ldots q_n}_{1;m_1;\ldots\ldots m_n}[\chi_3]}{m\frac{\Gamma(\alpha_1+\frac{1}{m}-1)\ \Gamma(\beta_1)}{\Gamma(\alpha_1+\beta_1+\frac{1}{m}-1)}\sum_{k_1=0}^{(\frac{V_1}{U_1})}\ldots\sum_{k_r=0}^{(\frac{V_r}{U_r})}(-v_r)_{u_1k_1\ldots}(-v_r)_{v_rk_r}A(v_1,k_1;\ldots v_r,k_r\prod_{i=1}^{r}\frac{y_ik_i}{k_1!}\times F^{p+\lambda:q_1;\ldots\ldots q_n}_{1+\lambda;\,m_1;\ldots\ldots m_n}[\chi_4]}$$

= 0 elsewhere (2.2)

Where

$$\chi_3=\left[\begin{matrix}(a_p):(b_{q_1});\ldots\ldots\ \ldots;(b_{q_n}^{(n)}); \\ (\alpha_1):(\beta_{m_1});\ldots\ldots\ldots;(\beta_{m_n}^{(n)});\end{matrix}\ z_1x^{m\lambda}(1-x^m)^{\lambda},\ldots,z_n^{\lambda}x^{m\lambda}(1-x^m)^{\lambda}\right]$$ and

$$\chi_4 = \left(\begin{array}{l} (a_p) : \frac{\alpha_1 + \frac{1}{m} - 1}{\lambda}, \ldots\ldots, \frac{(\alpha_1 + \lambda + \frac{1}{m} - 2)}{\lambda} : (b_{q_1});\ldots\ldots; (b_{q_n}^{(n)}); \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z_1, \ldots\ldots, z_n \\ (\alpha_1) : \frac{(\alpha_1 + \beta_1 + \frac{1}{m} - 1)}{\lambda}, \ldots\ldots, \frac{(\alpha_1 + \beta_1 + \lambda + \frac{1}{m} - 2)}{\lambda} ; (\beta_{m_1}); \ldots; (\beta_{m_n}^{(n)}); \end{array} \right)$$

Where $S_{V_1;\ldots\ldots V_r}^{U_1;\ldots\ldots U_r}[x_1\ldots x_r]$ be the multivariable polynomial which was formed by generalized the general class of polynomial as defined below

$$S_{V_1 \ldots\ldots V_r}^{U_1 \ldots\ldots U_r}[x_1, \ldots\ldots, x_r] = \sum_{k_1=0}^{(\frac{V_1}{U_1})} \sum_{k_r=0}^{(\frac{V_r}{U_r})} (-V_1)_{U_1 k_1 \ldots\ldots} (-V_r)_{U_r k_r + \ldots\ldots U_1 k} A(V, k_1 + \ldots . k_r)$$

$$\frac{x_1^{k_1}}{k_1!} \ldots\ldots \frac{x_r^{k_r}}{k_r!} \tag{2.3}$$

Where Vi = 0, 1,2 …….. (i = 1, …….., r), U_1,…….Ur are arbitrary positive integers, The coefficient A(V_1, k_1; ……..; V_r, k_r) being arbitrary constants, real or complex.

So by using (2.3) and by using the multivariable Hypergeometric function defined in (1.9), we occurred (2.1) then make its pdf form as (2.2).

Where

$$\alpha_1 = a + \sum_{j=1}^{n} \mu_j k_j \qquad \text{and} \qquad \beta_1 = b + \sum_{j=1}^{n} v_j k_j$$

Now assumed the following condition to be satisfied.

(vii) Re (a, b, r) › 0, Re (α_1, β_1) › 0

(viii) Min (λ, p, q) $\geq$ 0 (j = 1, 2, ……,n) (not all zero simultaneously)

(ix) (a) $\Delta_1 > 0,\ \Delta_2 > 0$

(b) $\Delta_1 + [V/U] > 0,\ \Delta_2 + [V/U] > 0$

(c) $\Delta_1 > 0,\ \Delta_2 + [V/U] > 0$

(d) $\Delta_1 + [V/U] > 0,\ \Delta_2 > 0$

Where

$$\Delta_1 = \operatorname{Re}(a) + p + \sum_{j=1}^{n} q_j \quad \text{and} \quad \Delta_2 = \operatorname{Re}(a) + p + \sum_{j=1}^{n} q_j$$

3.1 Theorem:3 Eulerian third Integral Involving $S_V^{U_1;\dots\dots;U_r}$ $[x_1\dots\dots x_r]$:

Prove that:

$$\int_0^1 x^{m(a-1)}(1-x^m)^{b-1} S_V^{U_1;\dots\dots;U_r}\left[y_1 x^{m\mu_1}(1-x^m)^{v_1},\dots\dots,\ y_r x^{m\mu_r}(1-x^m)^{v_r}\right]$$

$$\times F^{p:q_1;\dots\dots;q_n}_{1;m_1;\dots\dots;m_n}\left(\begin{matrix}(a_p):(b_{q_1});\dots\dots\ \dots\dots;(b_{q_n}^{(n)});\\(\alpha_1):(\beta_{m_1});\dots\dots\dots\dots;(\beta_{m_n}^{(n)});\end{matrix}\ z_1 x^{m\lambda},\dots\dots,\ z_n x^{m\lambda}\right) dx$$

$$= m\frac{\Gamma(\alpha_2 + \frac{1}{m} - 1)\,\Gamma(\beta_2)}{\Gamma(\alpha_2 + \beta_2 + \frac{1}{m} - 1)} \sum_{k_1,\dots\dots,\ k_r=0}^{U_1k_1+\dots\dots U_rk_r\le v} (-v)_{U_1K_1+\dots\dots U_rK_r}\, A(V_1, k_1;\dots\dots, k_r)$$

$$F^{p+\lambda:q_1;\ldots\ldots\ldots q_n}_{1+\lambda;m_1;\ldots\ldots\ldots m_n}$$

$$\times\left(\begin{matrix}(a_p):\dfrac{\alpha_2+\frac{1}{m}-1}{\lambda},\ldots\ldots,\dfrac{(\alpha_2+\lambda+\frac{1}{m}-2)}{\lambda}: & (b_{q_1});\ldots.;(b_{q_n}^{(n)}); \\ (\alpha_1):\dfrac{(\alpha_2+\beta_2+\frac{1}{m}-1)}{\lambda},\ldots\ldots,\dfrac{(\alpha_2+\beta_2+\lambda+\frac{1}{m}-2)}{\lambda} & (\beta_{m_1});\ldots;(\beta_{m_n}^{(n)});\end{matrix}\; z_1,\ldots,z_n\right)$$

(3.1)

Where $\alpha_2 = a + \sum_{j=1}^{n} \mu_j k_j$ and $\beta_2 = b + \sum_{j=1}^{n} v_j k_j$

Then p.d.f form of (3.1)

$$F(x) = \frac{x^{m(a-1)}(1-x^m)^{b-1} S_v^{u_1;\ldots\ldots v_r}[y_1 x^{mu}(1-x^m)^{v_1}\ldots.\, y_r x^{mu_r}(1-x^m)^{v_r}]F^{p:q_1;\ldots\ldots q_n}_{1;m_1;\ldots\ldots m_n}[\chi_5]}{m\dfrac{\Gamma(\alpha_2+\frac{1}{m}-1)\,\Gamma(\beta_2)}{\Gamma(\alpha_2+\beta_2+\frac{1}{m}-1)}\sum_{k_1,\ldots\ldots,k_r=0}^{U_1k_1+\ldots\ldots U_rk_r\le v}(-v_r)_{u_1k_1\ldots\ldots v_rk_r}A(v_1,k_1;\ldots v_r,k_r)F^{p+\lambda:q_1;\ldots\ldots q_n}_{1+\lambda;m_1;\ldots\ldots m_2}[\chi_6]}$$

= **0 elsewhere** (3.2)

Where

$$\chi_5 = \left[\begin{matrix}(a_p):(b_{q_1});\ldots\ldots\ldots.;(b_{q_n}^{(n)}); \\ (\alpha_1):(\beta_{m_1});\ldots\ldots\ldots;(\beta_{q_n}^{(n)});\end{matrix}\; z_1x^{m\lambda},\ldots.,z_nx^{m\lambda}\right]$$ and

$$\chi_6 = \left(\begin{array}{l} (a_p) : \dfrac{\alpha_2 + \frac{1}{m} - 1}{\lambda}, \ldots, \dfrac{(\alpha_2 + \frac{1}{m} + \lambda - 2)}{\lambda} : (b_{q_1}); \ldots; (b_{q_n}^{(n)}); \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z_1, \ldots, z_n \\ (\alpha_1) : \dfrac{(\alpha_2 + \beta_2 + \frac{1}{m} - 1)}{\lambda}, \ldots, \dfrac{(\alpha_2 + \beta_2 + \lambda + \frac{1}{m} - 2)}{\lambda}; (\beta_{m_1}); \ldots; (\beta_{m_n}^{(n)}); \end{array} \right)$$

Where $S_V^{U_1;\ldots\ldots;U_r}[x_1 \ldots x_r]$ be the multivariable polynomial which was formed by generalized the general class of polynomial as defined below

$$S_V^{U_1 \ldots\ldots U_r}[x_1, \ldots\ldots\ldots, x_r] = \sum_{k_1,\ldots\ldots k_r = 0}^{U_1 k_1 + \ldots U_r k_r \le V} (-V)_{U_1 k_1 + \ldots\ldots U_1 k} A(V, k_1 + \ldots k_r)$$

$$\times \frac{x_1^{k_1}}{k_1!} \ldots\ldots \frac{x_r^{k_r}}{k_r!}. \tag{3.3}$$

Where $U_1 \ldots\ldots\ldots U_r$ are arbitrary positive integers and the coefficient $A(V; k_1, \ldots k_r)$ $(V, k_1 \geq 0, \; i = 1, \ldots, r)$ are arbitrary constants, real or complex.

So by using (3.3) and the multivariable Hypergeometric function defined in (1.9) we occurred (3.1) and its pdf form (3.2).

Now assumed the following condition to be satisfied.

(xiii) $\text{Re}(a, b, r) > 0$, $\text{Re}(\alpha_2, \beta_2) > 0$

(xiv) $\text{Min}(\lambda, p, q) \geq 0$ $(j = 1, 2, \ldots, n)$ (not all zero simultaneously)

(xv) (a) $\Delta_1 > 0$, $\Delta_2 > 0$

(b) $\Delta_1 + [V/U] > 0$, $\Delta_2 + [V/U] > 0$

(c) $\Delta_1 > 0$, $\Delta_2 + [V/U] > 0$

(d) $\Delta_1 + [V/U] > 0$ and $\Delta_2 > 0$, where

$$\Delta_1 = [\ \mathrm{Re}\,(a) + p + \left(\sum_{j=1}^{n} q_j\right)]\quad \text{and} \quad \Delta_2 = \mathrm{Re}\,(a) + p + \sum_{j=1}^{n} q_j .$$

Proofs of Theorem 2 and Theorem 3: The equations (2.1) of theorem 2 and (3.1) of theorem 3, can be derived by following the same lines as given in the proof of first integral and making appropriate modifications and changes.

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