THE RULE OF CYCLE LENGTH AND GLOBAL ASYMPTOTIC STABILITY FOR A FOURTH-ORDER NONLINEAR DIFFERENCE EQUATION

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ABSTRACT

The rule of trajectory structure for fourth-order nonlinear difference equation

$$x_{n+1}=\frac{x_{n-2}^{b}x_{n-3}+1}{x_{n-2}^{b}+x_{n-3}},n=0,1,2,3,\cdots,$$

where $b\in[0,1)$ and the initial values $x_{-3},x_{-2},x_{-1},x_{0}\in(0,\infty)$, is described clearly out in this paper. Mainly, the lengths of positive and negative semi-cycles of its nontrivial solutions are found to occur periodically with prime period 15. The rule is $4^{-},3^{+},1^{1},2^{+},2^{-},1^{+},1^{-},1^{+}$ in a period. By utilizing this rule its positive equilibrium point is verified to be globally asymptotically stable.
Keywords: Rational difference equation; Trajectory structure rule; Monotonicity; Periodicity; Global asymptotic stability; semi-cycle length.

1. INTRODUCTION

In this paper we consider the following fourth-order nonlinear difference equation

$$x_{n+1}=\frac{x_{n-2}^{b}x_{n-3}+1}{x_{n-2}^{b}+x_{n-3}},n=0,1,2,3,\cdots, \quad (1)$$

where $b\in[0,1)$ and the initial values $x_{-3},x_{-2},x_{-1},x_{0}\in(0,\infty)$.

If $b=0$, then Eq.(1) will turn into the trivial case

$x_{n+1}=1,n=0,1,\cdots$. So，in what follows, we hypothesize that $0<b<1$.

when $b\in(0,1)$,then Eq(1)can be described as a nonlinear equation instead of a rational difference equation one .Unfortunately, we have not found the effective solution to the global behavior of nonlinear difference equations of order greater than one. Therefore, to study the qualitative properties of nonlinear difference equations with higher order is theoretically meaningful.

In this paper, it is vital for us to make sure the lengths of positive and negative semi-cycles of nontrivial solutions of Eq.(1) occur periodically with prime period 15 and the rule $4^{-},3^{+},1^{-},2^{+},2^{-},1^{+},1^{-},1^{+}$ in a period. Benefiting from the above rule and utilizing the monotonicity of solution, the positive equilibrium point of the equation is proved stable globally asymptotically.

Essentially, we come to the following conclusion for conclusion of Eq(1).

Theorem CL The rule of trajectory structure of Eq（1） is that none of its solutions do not asymptotically approach its equilibrium， furthermore， any

One of its solutions is either

(1) eventually trivial; or

(2) non-oscillatory and eventually positive(i.e., $x_n \geq 1$);or

(3) strictly oscillatory with the lengths of positive and negative semi-cycles periodically successively occurring with prime period 15 and the rule to be $4^-, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+$ in a period

Next, we will bear out the correctness of Theorem CL.

Obviously， the positive equilibrium $\bar{x}$ of Eq.(1) satisfies

$$\bar{x} = \frac{\bar{x}^b + \bar{x}}{\bar{x}^b \bar{x} + 1},$$

according to that, we can find that Eq.(1) has a unique equilibrium $\bar{x} = 1$.

Next, we will introduce some major definitions referring to this paper.

DEFININTION1.1 A positive semi-cycle $\{x_n\}_{n=-3}^{\infty}$ of a solution of Eq.(1) consists of a "string" of terms $x_l, x_{l+1}, \cdots, x_m$, all greater than or equal to the equilibrium $\bar{x}$, with $l \geq -3$ and $m \leq \infty$ such that

$$\text{either } l = -3 \text{ or } l > -3 \text{ and } x_{l-1} < \bar{x}$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} < \bar{x}$$

A negative semi-cycle of a solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1) consists of a "string" of terms $x_l, x_{l+1}, \cdots, x_m$, all less than $\bar{x}$, with $l \geq -3$ and $m \leq \infty$ such that

$$\text{either } l = -3 \text{ or } l > -3 \text{ and } x_{l-1} \geq \bar{x}$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} \geq \bar{x}$$

The length of a semi-cycle is the number of the total terms contained in it.

DEFINTION 1.2 A solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1) is said to be eventually trivial if x_n is eventually equal to $\bar{x}=1$; Otherwise, the solution is said to be non-trivial. A solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1) is said to be eventually positive(negative) if x_n is eventually great(less) than $\bar{x}=1$.

For the other concepts in this paper and related work, see [3,4] and [1,2,5-10], respectively.

2. THREE LEMMAS

In order to improve the main result, we first establish three basic lemmas

LEMMA 2.1 A solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1) is eventually trivial if and only if

$$(x_{-3}-1)(x_{-2}-1)(x_{-1}-1)(x_0-1)=0. \quad 2.1$$

Proof. Sufficiency. Assume that (2.1) holds. Then it follows from Eq.(1) that the following conclusions hold.

i) if $x_{-3}-1=0$, then $x_n=1$ for $n\geq 7$;

ii) if $x_{-2}-1=0$, then $x_n=1$ for $n\geq 4$;

iii) if $x_{-1}-1=0$, then $x_n=1$ for $n\geq 5$;

iv) if $x_0-1=0$, then $x_n=1$ for $n\geq 6$;

Necessity. Conversely, assume that

$$(x_{-3}-1)(x_{-2}-1)(x_{-1}-1)(x_0-1)\neq 0. \quad (2.2)$$

Then one can show that

$$x_n\neq 1 \text{ for any } n\geq 1.$$

Assume the contrary that for some $N\geq 1$

$$x_N=1 \text{ and that } x_n\neq 1 \text{ for } -3\leq n\leq N-1. \quad (2.3)$$

Obviously,

$$1=x_N=\frac{x_{N-3}^{b}x_{N-4}+1}{x_{N-3}^{b}+x_{N-4}},$$

And it means

$(x_{N-3}^{b}-1)(x_{N-4}-1)=0$, which contradicts (2.3).

REMARK 2.1 Lemma 2.1 actually demonstrates that a solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1) is eventually nontrivial if and only if

$$(x_{-3}-1)(x_{-2}-1)(x_{-1}-1)(x_0-1)\neq 0.$$

Therefore, if a solution $\{x_n\}_{n=-3}^{\infty}$ is nontrivial, then $x_n \neq 1$ for $n \geq -3$.

LEMMA 2.2 Let $\{x_n\}_{n=-3}^{\infty}$ be a nontrivial positive solution of Eq.(1). Then the following conclusions are true:

(a) $(x_{n+1}-1)(x_{n-2}-1)(x_{n-3}-1) > 0$ for $n \geq 0$;

(b) $(x_{n-1}-x_{n-2})(x_{n-2}-1) < 0$ for $n \geq 0$.

Proof In view of Eq.(1), we can see that

$$x_{n+1}-1=\frac{(x_{n-2}^{b}-1)(x_{n-3}-1)}{x_{n-2}^{b}+x_{n-3}}, n=0,1,\cdots$$

and

$$x_{n+1}-x_{n-3}=\frac{(1-x_{n-3})(1+x_{n-3})}{x_{n-2}^{b}+x_{n-3}}, n=0,1,\cdots.$$

From which inequalities (a) and (b) follow. So the proof is complete.

LEMMA 2.3 There exist non-oscillatory solutions of Eq.(1), which must be eventually positive. There don't exist eventually negative non-oscillatory solutions of Eq.(1).

Proof Consider a solution of Eq.(1) with $x_{-3} > 1, x_{-2} > 1, x_{-1} > 1$ and $x_0 > 1$.We then know from Lemma 2.2(a) that $x_n > 1$ for $n \geq -3$. So, this solution is just a non-oscillatory solution and furthermore eventually positive.

So does a positive integer N such that $x_n < 1$ for $n > N$. There out, for $n \geq N+3, (x_{n+1}-1)(x_{n-2}-1)(x_{n-3}-1) < 0$ .This contradicts Lemma 2.2(a). So, the possibility of existence of eventually negative non-oscillatory solutions of Eq.(1) is zero.

3. MAIN RESULTS AND THEIR PROOFS

First we analyze the structure of the semi-cycles of nontrivial solutions of Eq.(1). Here we confine us to consider the situation of the strictly oscillatory solution of Eq.(1).

THEOREM 3.1 Let $\{x_n\}_{n=-3}^{\infty}$ be any strictly oscillatory solution of Eq.(1). Then, the lengths of positive and negative semi-cycles of the solution periodically successively occur with prime period 15. And in a period, the rule is $4^-, 3^+, 1^1, 2^+, 2^-, 1^+, 1^-, 1^+$.

Proof By Lemma 2.2(a), one can see that the length of a negative semi-cycle is not larger than 4, whereas, the length of a positive semi-cycle is 3 at most. Based on the strictly oscillatory character of the solution, we see, for some integer $p \geq 0$, one of the following four cases must occur:

Case 1: $x_{p-3} < 1, x_{p-2} > 1, x_{p-1} < 1, x_p < 1;$

Case 2: $x_{p-3} < 1, x_{p-2} > 1, x_{p-1} < 1, x_p > 1;$

Case 3: $x_{p-3} < 1, x_{p-2} > 1, x_{p-1} > 1, x_p < 1;$

Case 4: $x_{p-3} < 1, x_{p-2} > 1, x_{p-1} > 1, x_p > 1.$

If Case 1 occurs, it follows from Lemma 2.2(a) that

$x_{p-3} < 1, x_{p-2} > 1, x_{p-1} < 1, x_p < 1, x_{p+1} < 1, x_{p+2} < 1, x_{p+3} > 1, x_{p+4} > 1, x_{p+5} > 1, x_{p+6} < 1, x_{p+7} > 1,$
$x_{p+8} > 1, x_{p+9} < 1, x_{p+10} < 1, x_{p+11} > 1, x_{p+12} < 1, x_{p+13} > 1, x_{p+14} < 1, x_{p+15} < 1, x_{p+16} < 1, x_{p+17} < 1,$
$x_{p+18} > 1, x_{p+19} > 1, x_{p+20} > 1, x_{p+21} < 1, x_{p+22} > 1, x_{p+23} > 1, x_{p+24} < 1, x_{p+25} < 1, x_{p+26} > 1, x_{p+27} < 1,$ which
$x_{p+28} > 1, x_{p+29} < 1, x_{p+30} < 1, x_{p+31} < 1, x_{p+32} < 1, x_{p+33} > 1, x_{p+34} > 1, x_{p+35} > 1, x_{p+36} < 1, x_{p+37} > 1,$
$x_{p+38} > 1, x_{p+39} < 1, x_{p+40} < 1, x_{p+41} > 1, \cdots,$

means that the rule for the lengths of positive and negative semi-cycles of the solution of Eq.(1) to successively occur is $\cdots, 4^-, 3^+, 1^1, 2^+, 2^-, 1^+, 1^-, 1^+, \cdots.$

If Case 2 happens, then Lemma 2.2(a) tells us that

$x_{p-3} < 1, x_{p-2} > 1, x_{p-1} < 1, x_p > 1, x_{p+1} < 1, x_{p+2} < 1, x_{p+3} < 1, x_{p+4} < 1, x_{p+5} > 1, x_{p+6} > 1, x_{p+7} > 1,$
$x_{p+8} < 1, x_{p+9} > 1, x_{p+10} > 1, x_{p+11} < 1, x_{p+12} < 1, x_{p+13} > 1, x_{p+14} < 1, x_{p+15} > 1, x_{p+16} < 1, x_{p+17} < 1,$
$x_{p+18} < 1, x_{p+19} < 1, x_{p+20} > 1, x_{p+21} > 1, x_{p+22} > 1, x_{p+23} < 1, x_{p+24} > 1, x_{p+25} > 1, x_{p+26} < 1, x_{p+27} < 1,$ which
$x_{p+28} > 1, x_{p+29} < 1, x_{p+30} > 1, x_{p+31} < 1, x_{p+32} < 1, x_{p+33} < 1, x_{p+34} < 1, x_{p+35} > 1, x_{p+36} > 1, x_{p+37} > 1,$
$x_{p+38} < 1, x_{p+39} > 1, x_{p+40} > 1, x_{p+41} < 1, \cdots,$

means that the rule for the lengths of positive and negative semi-cycles of the solution of Eq.(1) to

successively occur is $\cdots, 4^-, 3^+, 1^1, 2^+, 2^-, 1^+, 1^-, 1^+, \cdots.$

This shows the rule for the numbers of terms of positive and negative semi-cycles of the solution of Eq.(1) to successively occur still is $\cdots, 4^-, 3^+, 1^1, 2^+, 2^-, 1^+, 1^-, 1^+, \cdots$

If Case 3 happens, then Lemma 2.2(a) implies that

$x_{p-3} < 1, x_{p-2} > 1, x_{p-1} > 1, x_p < 1, x_{p+1} < 1, x_{p+2} > 1, x_{p+3} < 1, x_{p+4} > 1, x_{p+5} < 1, x_{p+6} < 1, x_{p+7} < 1,$
$x_{p+8} < 1, x_{p+9} > 1, x_{p+10} > 1, x_{p+11} > 1, x_{p+12} < 1, x_{p+13} > 1, x_{p+14} > 1, x_{p+15} < 1, x_{p+16} < 1, x_{p+17} > 1,$
$x_{p+18} < 1, x_{p+19} > 1, x_{p+20} < 1, x_{p+21} < 1, x_{p+22} < 1, x_{p+23} < 1, x_{p+24} > 1, x_{p+25} > 1, x_{p+26} > 1, x_{p+27} < 1,$
$x_{p+28} > 1, x_{p+29} > 1, x_{p+30} < 1, x_{p+31} < 1, x_{p+32} > 1, x_{p+33} < 1, x_{p+34} > 1, x_{p+35} < 1, x_{p+36} < 1, x_{p+37} < 1,$
$x_{p+38} < 1, x_{p+39} > 1, x_{p+40} > 1, x_{p+41} > 1, \cdots,$

This shows the rule for the numbers of terms of positive and negative semi-cycles of the solution of Eq.(1) to successively occur still is $\cdots, 4^-, 3^+, 1^1, 2^+, 2^-, 1^+, 1^-, 1^+, \cdots$

If Case 4 happens, then it is to see from Lemma2.2 (a) that

$$x_{p-3}<1,x_{p-2}>1,x_{p-1}>1,x_{p}>1,x_{p+1}<1,x_{p+2}>1,x_{p+3}>1,x_{p+4}<1,x_{p+5}<1,x_{p+6}>1,x_{p+7}<1,$$
$$x_{p+8}>1,x_{p+9}<1,x_{p+10}<1,x_{p+11}<1,x_{p+12}<1,x_{p+13}>1,x_{p+14}>1,x_{p+15}>1,x_{p+16}<1,x_{p+17}>1,$$
$$x_{p+18}>1,x_{p+19}<1,x_{p+20}<1,x_{p+21}>1,x_{p+22}<1,x_{p+23}>1,x_{p+24}<1,x_{p+25}<1,x_{p+26}<1,x_{p+27}<1,$$
$$x_{p+28}>1,x_{p+29}>1,x_{p+30}>1,x_{p+31}<1,x_{p+32}>1,x_{p+33}>1,x_{p+34}<1,x_{p+35}<1,x_{p+36}>1,x_{p+37}<1,$$
$$x_{p+38}>1,x_{p+39}<1,x_{p+40}<1,x_{p+41}<1,\cdots,$$

This shows the rule for the numbers of terms of positive and negative semi-cycles of the solution of Eq.(1) to successively occur still is $\cdots,4^{-},3^{+},1^{1},2^{+},2^{-},1^{+},1^{-},1^{+},\cdots$

Hence, the proof is complete.

Now, we present the global asymptotical stable results for Eq.(1).

THEOREM 3.2 Assume that $a\in[0,1)$. Then the unique positive equilibrium of Eq.(1) is globally asymptotically stable.

Proof When $b=0$, Eq.(1) is trivial. So, we only consider the case $b>0$, and prove that the positive equilibrium point $\bar{x}$ of Eq.(1) is both locally asymptotically stable and globally attractive. The literalized equation of Eq.(1) about the positive equilibrium $\bar{x}=1$ is

$$y_{n+1}=0\cdot y_{n}+0\cdot y_{n-1}+0\cdot y_{n-2}+0\cdot y_{n-3},n=0,1,\cdots.$$

By virtue of [4, Remark 1.3.1, P13], $\bar{x}$ is locally asymptotically stable. It remains to be verified that every positive solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1) converges to $\bar{x}$ as $n\to\infty$.Namely, we want to prove

$$\lim_{n\to\infty}x_n=\bar{x}=1. \tag{3.1}$$

If the initial values of the solutions satisfy (2.1),i.e. the solution is a trivial solution, then lemma2.1 says that the solution is eventually equal to 1 and of course, (3.1) holds.

If the solution is a non-trivial solution, then we can further divide the solution into two cases.

a) Non-oscillatory solution;
b) Oscillatory solution.

If case a) happens, then it follows from Lemma 2.2 that the solution is actually an eventually positive one. According to Lemma2.2 (b), we see that x_{4n},x_{4n-1},x_{4n-2} and x_{4n-3} are eventually decreasing and bounded from the upper by the constant 1. So the limits

$$\lim_{n\to\infty}x_{4n}=G,\lim_{n\to\infty}x_{4n+1}=L,\lim_{n\to\infty}x_{4n+2}=M\ \text{ and }\ \lim_{n\to\infty}x_{4n+3}=N$$

Exist and are finite. Noting

$$x_{4n+1}=\frac{x_{4n-2}^{b}x_{4n-3}+1}{x_{4n-2}^{b}x_{4n-3}},{}_{4n}=\frac{x_{4n-3}^{b}x_{4n-4}+1}{x_{4n-3}^{b}x_{4n-4}},{}_{4n+2}=\frac{x_{4n-1}^{b}x_{4n-2}+1}{x_{4n-1}^{b}x_{4n-2}},\text{and }{}_{4n+3}=\frac{x_{4n}^{b}x_{4n-1}+1}{x_{4n}^{b}x_{4n-1}},$$

and taking the limits on both sides of the above equalities, respectively, one may obtain

$$L = \frac{M^b L + 1}{M^b + L}, G = \frac{L^b G + 1}{L^b + G}, M = \frac{N^b M + 1}{N^b + M}, \text{ and } N = \frac{G^b N + 1}{G^b + N}.$$

Solving these equations, we get $G = L = M = N = 1$, which shows (3.1) is true.

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