CODING THEOREMS FOR THE R-NORM INFORMATION MEASURE

***Dr Meenakshi Darolia**
Assistant Professor, Isharjyot Degree College Pehova
****Mr Vishesh Bansal**
Assistant Professor, MAIMT Jagadhri

In this paper, we will give coding theorems with respect to the R-Norm Information Measure. Suppose we have discrete memory less information source with an encoding alphabet with D Symbols and code words x_i with word-lengths N_i, i = 1,2,….n, which fulfill the Kraft inequality

$$\left(\sum_{i=1}^{n} D^{-n_i}\right) \leq 1 \qquad (1.1)$$

Where D is the size of the code alphabet.

We next give a definition of average length L_R of cord words.

DEFINITION: The average length L_R with respect to R-norm information measure is for R€ R^+ given by

$$L_R = \frac{R}{R-1}\left[1-\sum_{i=1}^{n} P_i D^{-n_i(R-1)/R}\right]$$

Clearly L_R will increase for increasing word lengths. An important property of L_R is that for R→1 it is equivalent with the average length of code words by Shannon, up to a constant.

Theorem: For all integer D > 1

$$L_R = \frac{R}{R-1}\left[1-\sum_{i=1}^{n} P_i D^{-n_i(R-1)/R}\right] > 0 \quad \text{for} \quad R€ R^+$$

Proof: To prove $L_R > 0$, we consider the following cases:

Case I: when R > 1, then R – 1 > 0 and $\frac{R}{R-1} > 0$

From (1.1), we have $\left(\sum_{i=1}^{n} D^{-n_i}\right) \le 1 \quad \Rightarrow D^{-n_i} < 1 \quad \Rightarrow D^{-n_i\left(\frac{R}{R-1}\right)} < 1$

Multiplying both sides of (1.3) by P_i, we get $\Rightarrow P_i D^{-n_i\left(\frac{R}{R-1}\right)} < P_i$

Summing over $i = 1,2,3,\ldots\ldots\ldots,N$ both sides, we get

$$\Rightarrow \sum_{i=1}^{n} P_i D^{-n_i\left(\frac{R}{R-1}\right)} < \sum_{i=1}^{n} P_i = 1, \qquad \Rightarrow \sum_{i=1}^{n} P_i D^{-N_i\left(\frac{R}{R-1}\right)} < 1$$

$$\Rightarrow -\sum_{i=1}^{n} P_i D^{-N_i\left(\frac{R}{R-1}\right)} > -1, \qquad \Rightarrow 1-\sum_{i=1}^{n} P_i D^{-N_i\left(\frac{R}{R-1}\right)} > -1+1 = 0$$

$$\Rightarrow 1-\sum_{i=1}^{n} P_i D^{-n_i\left(\frac{R}{R-1}\right)} > 0 \qquad (1.4)$$

Multiplying (1.1) by $\frac{R}{R-1}$, we get

$$\frac{R}{R-1}\left[1-\sum_{i=1}^{n} P_i D^{-\left(n_i\left(R-1\right)/R\right)}\right] > 0 \qquad (1.5)$$

But $L_R = \frac{R}{R-1}\left[1-\sum_{i=1}^{n} P_i D^{-\left(N_I\left(R-1\right)/R\right)}\right]$

Thus from (1.5), we get

$$L_R = \frac{R}{R-1}\left[1-\sum_{i=1}^{n} P_i D^{-\left(N_I\left(R-1\right)/R\right)}\right] > 0 \quad \text{for } R > 1 \qquad (1.6)$$

Case II: when 0 < R < 1 ⇒ R – 1 < 0 and $\frac{R}{R-1} < 0$

From (1.1), we have

$$\left(\sum_{i=1}^{n} D^{-N_i}\right) \leq 1, \quad \Rightarrow D^{-N_i} < 1, \quad \Rightarrow D^{-N_i\left(\frac{R}{R-1}\right)} > 1 \tag{1.7}$$

Multiplying both sides of (1.7) by P_i , we get

$$\Rightarrow P_i D^{-N_i\left(\frac{R}{R-1}\right)} > P_i$$

Summing over $i = 1,2,3,\ldots\ldots\ldots,n$ both sides, we get

$$\Rightarrow \sum_{i=1}^{n} P_i D^{-N_i\left(\frac{R}{R-1}\right)} > \sum_{i=1}^{n} P_i = 1$$

$$\Rightarrow \sum_{i=1}^{n} P_i D^{-N_i\left(\frac{R}{R-1}\right)} > 1$$

$$\Rightarrow -\sum_{i=1}^{n} P_i D^{-N_i\left(\frac{R}{R-1}\right)} < -1$$

$$\Rightarrow 1-\sum_{i=1}^{n} P_i D^{-N_i\left(\frac{R}{R-1}\right)} < -1+1=0$$

$$\Rightarrow 1-\sum_{i=1}^{n} P_i D^{-N_i\left(\frac{R}{R-1}\right)} < 0 \tag{1.8}$$

Multiplying (1.8) by $\frac{R}{R-1}$, we get

$$\frac{R}{R-1}\left[1-\sum_{i=1}^{n} P_i D^{-N_i (R-1)/R}\right] > 0 \tag{1.9}$$

But $L_R = \frac{R}{R-1}\left[1-\sum_{i=1}^{n} P_i D^{-N_i (R-1)/R}\right]$

Thus from (1.9) , we get

$L_R = \frac{R}{R-1}\left[1-\sum_{i=1}^{n} P_i D^{-N_i (R-1)/R}\right] > 0$ for $0 < R < 1$ (1.10)

Thus from (1.6) and (1.10), we get

$L_R = \frac{R}{R-1}\left[1-\sum_{i=1}^{n} P_i D^{-n_I (R-1)/R}\right] > 0$ for $R \in R^+$

Theorem: If N_i , i=1,2,........,n are in length of code words x_i then

$$\lim_{R\to 1} L_R = \sum_{i=1}^{n} p_i N_i \log(D)$$

Proof: The average length L_R with respect to R-norm information measure is for $R \in R^+$ given by

$L_R = \frac{R}{R-1}\left[1-\sum_{i=1}^{n} P_i D^{-N_I (R-1)/R}\right]$ (1.11)

Taking limit both sides as R→1, we get

$\lim_{R\to 1} L_R = \lim_{R\to 1} \frac{R}{R-1}\left[1-\sum_{i=1}^{n} P_i D^{-N_I (R-1)/R}\right] = \frac{0}{0}(\,form)$ (1.12)

Thus by Bernoulli-L' Hospital theorem, we get

$\lim_{R\to 1} L_R = \lim_{R\to 1}\left[1.\left[1-\sum_{i=1}^{n} p_i D^{-N_I (R-1)/R}\right] - R\left[0-\sum_{i=1}^{n} p_i \frac{dT}{dR}\right]\right]$ (1.13)

Where T = $D^{-N_i (R-1)/R}$ (1.11)

Taking log both sides of (1.11), we get

$\log T = \frac{-N_i (R-1)}{R} \log D$ (1.15)

Diff w.r.t ‘R’ both sides of (1.15), we get

$$\frac{1}{T}\cdot\frac{dT}{dR} = -N_i \log(D)\left[\frac{R.1-1.(R-1)}{R^2}\right]$$

$$\frac{dT}{dR} = -TN_i \log(D)\left[\frac{1}{R^2}\right] = -\frac{1}{R^2} D^{-N_i(R-1)/R}[N_i \log(D)]. \qquad (1.16)$$

Substitute (1.16) in (1.13), we get

$$\Rightarrow \lim_{R\to 1} L_R = \lim_{R\to 1}\left[1.\left[1-\sum_{i=1}^{n} p_i D^{-N_i(R-1)/R}\right] - \frac{R}{R^2}\left[\sum_{i=1}^{n} p_i D^{-N_i(R-1)/R} N_i \log(D)\right]\right]$$

$$\Rightarrow \lim_{R\to 1} L_R = \lim_{R\to 1}\left[\left[1-\sum_{i=1}^{n} p_i D^{-N_i(R-1)/R}\right] - \frac{1}{R}\left[\sum_{i=1}^{n} p_i D^{-N_i(R-1)/R} N_i \log(D)\right]\right]$$

$$\Rightarrow \lim_{R\to 1} L_R = \left[\left[1-\sum_{i=1}^{n} p_i\right] - \left[\sum_{i=1}^{n} p_i [N_i \log(D)]\right]\right] = -\sum_{i=1}^{n} p_1 N_i \log(D) \qquad (1.17)$$

Thus finally $\lim_{R\to 1} L_R = -\sum_{i=1}^{n} p_1 N_i \log(D)$ (1.18)

Theorem: For all integer D > 1

$$H_R(P) \le L_R$$

Under the condition (1.1). Equality holds if and only if

$$N_i = -\log(P_i^R / \sum_{i=1}^{n} P_i^R)$$

Proof: To prove this theorem, we consider following cases:

Case I: when R > 1

We use Holder inequality [12]

$$\sum_{i=1}^{n} x_i y_i \ge \left(\sum_{i=1}^{n} x_i^P\right)^{\frac{1}{P}}\left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}} \qquad (1.19)$$

for all $x_i \geq 0, y_i \geq 0$, i = 1,2,....n when P <1(≠) and $p^{-1}+q^{-1}=1$ .with equality if and only if there exists a positive number c such that

$x_i^{\,p} = cy_i^{\,q}$ Setting $x_i = P_i^{\frac{R}{R-1}} . D^{-N_i}$

and $y_i = P_i^{\frac{R}{R-1}}$, P = $1-\frac{1}{R}$, q = $1-R$

$$\left(\sum_{i=1}^{n} P_i^{\frac{R}{R-1}} D^{-N_i} P_i^{\frac{R}{R-1}}\right) \geq \left(\sum_{i=1}^{n} (P_i^{\frac{R}{R-1}} D^{-N_i})^{1-\frac{1}{R}}\right)^{1/1-\frac{1}{R}} \left(\sum_{i=1}^{n} (P_i^{\frac{R}{R-1}})^{1-R}\right)^{1/1-R} \quad (1.20)$$

$$\left(\sum_{i=1}^{n} D^{-N_i}\right) \geq \left(\sum_{i=1}^{n} P_i \; (D^{-N_i})^{\frac{R-1}{R}}\right)^{\frac{R}{R-1}} \left(\sum_{i=1}^{n} (P_i \;)^{R}\right)^{1/1-R} \quad (1.21)$$

Since $\left(\sum_{i=1}^{n} D^{-n_i}\right) \leq 1$ Thus (1.21) becomes

$$\left(\sum_{i=1}^{n} P_i \; (D^{-N_i})^{\frac{R-1}{R}}\right)^{\frac{R}{R-1}} \left(\sum_{i=1}^{n} (P_i \;)^{R}\right)^{1/1-R} \leq 1$$

$$\left(\sum_{i=1}^{n} P_i \; (D^{-N_i})^{\frac{R-1}{R}}\right)^{\frac{R}{R-1}} \leq \left(\sum_{i=1}^{n} (P_i \;)^{R}\right)^{1/R-1} \quad (1.22)$$

Raising power 1/R both sides of (1.22), we get

$$\left(\sum_{i=1}^{n} P_i \; (D^{-N_i})^{\frac{R-1}{R}}\right) \leq \left(\sum_{i=1}^{n} (P_i \;)^{R}\right)^{\frac{1}{R}}$$

$$\Rightarrow -\left(\sum_{i=1}^{n} P_i \; (D^{-N_i})^{\frac{R-1}{R}}\right) \geq -\left(\sum_{i=1}^{n} (P_i \;)^{R}\right)^{\frac{1}{R}}$$

$$\Rightarrow 1-\left(\sum_{i=1}^{n} P_i \ (D^{-N_i})^{\frac{R-1}{R}}\right) \geq 1-\left(\sum_{i=1}^{n} (P_i \ )^{R}\right)^{\frac{1}{R}} \quad (1.23)$$

We know $\dfrac{R}{R-1} > 0$ if R > 1

Multiplying $\dfrac{R}{R-1}$ by both side of (1.23) and we get

$$\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i D^{-n_i(\frac{R}{R-1})}\right)\right) \geq \frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i^{\,R}\right)^{\frac{1}{R}}\right) \quad (1.21)$$

But $$\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i^{\,R}\right)^{\frac{1}{R}}\right) = H_R(P)$$

and $$\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i D^{-n_i(\frac{R}{R-1})}\right)\right) = L_R$$

Thus from (1.21), we get $L_R \geq H_R(P)$ for R >1 (1.25)

Case II: when 0 < R < 1

We use Holder inequality [12]

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^{\,P}\right)^{\frac{1}{P}}\left(\sum_{i=1}^{n} y_i^{\,q}\right)^{\frac{1}{q}} \quad (1.26)$$

For all $x_i \geq 0, y_i \geq 0$, i=1,2,....N when P <1(≠) and $p^{-1} + q^{-1} = 1$.with equality if and only if there exists a positive number c such that

$x_i^{\,p} = c y_i^{\,q}$ Setting $x_i = P_i^{\frac{R}{R-1}} . D^{-n_i}$

and $y_i = P_i^{\frac{R}{R-1}}$, P = $1-\dfrac{1}{R}$, q = $1-R$ Thus (1.27) becomes

$$\left(\sum_{i=1}^{n} P_i^{\frac{R}{R-1}} D^{-N_i} P_i^{\frac{R}{R-1}}\right) \leq \left(\sum_{i=1}^{n} (P_i^{\frac{R}{R-1}} D^{-N_i})^{1-\frac{1}{R}}\right)^{1/1-\frac{1}{R}} \left(\sum_{i=1}^{n} (P_i^{\frac{R}{R-1}})^{1-R}\right)^{1/1-R}$$

$$\left(\sum_{i=1}^{n} D^{-N_i}\right) \leq \left(\sum_{i=1}^{n} P_i \ (D^{-N_i})^{\frac{R-1}{R}}\right)^{\frac{R}{R-1}} \left(\sum_{i=1}^{n} (P_i \ )^R\right)^{1/1-R} \qquad (1.27)$$

Since $\left(\sum_{i=1}^{n} D^{-n_i}\right) \leq 1$ Thus (1.27) becomes

$$\left(\sum_{i=1}^{n} P_i \ (D^{-N_i})^{\frac{R-1}{R}}\right)^{\frac{R}{R-1}} \left(\sum_{i=1}^{n} (P_i \ )^R\right)^{1/1-R} \geq 1$$

$$\Rightarrow \left(\sum_{i=1}^{n} P_i \ (D^{-N_i})^{\frac{R-1}{R}}\right)^{\frac{R}{R-1}} \geq \left(\sum_{i=1}^{n} (P_i \ )^R\right)^{1/R-1} \qquad (1.28)$$

Raising power 1/R both sides of (1.28), we have

$$\left(\sum_{i=1}^{n} P_i \ (D^{-N_i})^{\frac{R-1}{R}}\right) \geq \left(\sum_{i=1}^{n} (P_i \ )^R\right)^{\frac{1}{R}}$$

$$\Rightarrow -\left(\sum_{i=1}^{n} P_i \ (D^{-N_i})^{\frac{R-1}{R}}\right) \leq -\left(\sum_{i=1}^{n} (P_i \ )^R\right)^{\frac{1}{R}}$$

$$\Rightarrow 1-\left(\sum_{i=1}^{n} P_i \ (D^{-N_i})^{\frac{R-1}{R}}\right) \leq 1-\left(\sum_{i=1}^{n} (P_i \ )^R\right)^{\frac{1}{R}} \qquad (1.29)$$

We know $\frac{R}{R-1}$ < 0 if 0 < R < 1

Multiplying $\frac{R}{R-1}$ by both sides (1.29), we get

$$\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i D^{-N_i(\frac{R}{R-1})}\right)\right) \geq \frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i^{R}\right)^{\frac{1}{R}}\right) \quad (1.30)$$

But $\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i^{R}\right)^{\frac{1}{R}}\right) = H_R(P)$

and $\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i D^{-n_i(\frac{R}{R-1})}\right)\right) = L_R$ (1.31)

Thus from (1.30),we get $L_R \geq H_R(P)$ 0 < R < 1 (1.32)

Thus from (1.25) and (1.32), we get

$L_R \geq H_R(P)$ for R € R+ (1.33)

Theorem: For every code with length $n_i, i = 1,2,3,\ldots\ldots, N, and\ L_R$ made to satisfy

$$L_R < H_R(P).D^{\left(\frac{R-1}{R}\right)} + \frac{R}{R-1}\left[1-D^{\left(\frac{R-1}{R}\right)}\right] \quad \text{for } R \in R^+$$

Proof: To prove this theorem we consider the following cases:

Case I: when R > 1

Let n_i be the positive integer satisfying the inequality by (11)

$$-\log\left(\frac{p_i^R}{\sum p_i^R}\right) \leq n_i < -\log\left(\frac{p_i^R}{\sum p_i^R}\right)+1$$

Consider the interval

$\delta_i = \left[-\log\left(\frac{p_i^R}{\sum p_i^R}\right), -\log\left(\frac{p_i^R}{\sum p_i^R}\right)+1\right]$ of length 1 .In every δ_i ,there lies exactly one positive number n_i, such that

$$0 < -\log\left(\frac{p_i^R}{\sum p_i^R}\right) \le n_i < -\log\left(\frac{p_i^R}{\sum p_i^R}\right) + 1 \qquad (1.31)$$

It can be shown that the sequence $\{n_i\}, i = 1,2,3,......., N$ thus defined, satisfies (1.1).

Thus from (1.31), we get

$$n_i < -\log\left(\frac{p_i^R}{\sum p_i^R}\right) + 1$$

$$-\log D^{-n_i} < -\log\left[\left(\frac{p_i^R}{\sum p_i^R}\right) D^{-1}\right]$$

$$D^{-n_i} > \left[\frac{p_i^R}{\sum p_i^R}\right] D^{-1}$$

Raising above inequality by $\frac{R-1}{R}$ both sides, we get

$$D^{-n_i\left(\frac{R-1}{R}\right)} > \left(\frac{p_i^R}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)} \qquad (1.35)$$

Multiplying both sides of (1.35) by p_i , we get

$$P_i D^{-n_i\left(\frac{R-1}{R}\right)} > P_i\left(\frac{p_i^R}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow P_i D^{-n_i\left(\frac{R-1}{R}\right)} > P_i.(P_i)^{R-1}\left(\frac{1}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow P_i D^{-n_i\left(\frac{R-1}{R}\right)} > (P_i)^{1-1+R}\left(\frac{1}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow P_i D^{-n_i\left(\frac{R-1}{R}\right)} > (P_i)^{R}\left(\frac{1}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)} \qquad (1.36)$$

Summing over $i = 1,2,3,\ldots\ldots,N$ both sides of (1.36), we get

$$\Rightarrow \sum P_i.D^{-n_i\left(\frac{R-1}{R}\right)} > (\sum P_i^R)\left(\frac{1}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow \sum P_i.D^{-n_i\left(\frac{R-1}{R}\right)} > .(\sum P_i^R)^{1-1+\frac{1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow \sum P_i.D^{-n_i\left(\frac{R-1}{R}\right)} > .(\sum P_i^R)^{\frac{1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow -\sum P_i.D^{-n_i\left(\frac{R-1}{R}\right)} < -.(\sum P_i^R)^{\frac{1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow 1-\sum P_i.D^{-n_i\left(\frac{R-1}{R}\right)} < 1-.(\sum P_i^R)^{\frac{1}{R}} D^{\left(\frac{1-R}{R}\right)} \qquad (1.37)$$

We know $\frac{R}{R-1} > 0$ if R > 1

Multiplying $\frac{R}{R-1}$ by both sides of (1.37), we get

$$\Rightarrow \frac{R}{R-1}\left[1-\left(\sum P_i.D^{-n_i(\frac{R}{R-1})}\right)\right] < \frac{R}{R-1}\left[1-(\sum P_i^R)^{\frac{1}{R}}.D^{\left(\frac{1-R}{R}\right)}\right]$$

$$\Rightarrow \frac{R}{R-1}\left[1-\left(\sum P_i.D^{-n_i(\frac{R}{R-1})}\right)\right] < \frac{R}{R-1}-\frac{R}{R-1}\left[(\sum P_i^R)^{\frac{1}{R}}.D^{\left(\frac{1-R}{R}\right)}\right]$$

$$\Rightarrow \frac{R}{R-1}\left[1-\left(\sum P_i . D^{-n_i(\frac{R}{R-1})}\right)\right] < \frac{R}{R-1} - \frac{R}{R-1}\left[\left(\sum P_i^R\right)^{\frac{1}{R}} . D^{\left(\frac{1-R}{R}\right)}\right]$$

$$\Rightarrow \frac{R}{R-1}\left[1-\left(\sum P_i . D^{-n_i(\frac{R}{R-1})}\right)\right] < \frac{R}{R-1} - \frac{R}{R-1}\left[1-1+\left(\sum P_i^R\right)^{\frac{1}{R}}.\right] . D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow \frac{R}{R-1}\left[1-\left(\sum P_i . D^{-n_i(\frac{R}{R-1})}\right)\right] < \frac{R}{R-1}\left[1-D^{\left(\frac{1-R}{R}\right)}\right] + \frac{R}{R-1}\left[1-\left(\sum P_i^R\right)^{\frac{1}{R}}.\right] . D^{\left(\frac{1-R}{R}\right)}$$

(1.38)

But $\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{N} P_i^R\right)^{\frac{1}{R}}\right) = H_R(P)$ And $\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{N} P_i D^{-n_i(\frac{R}{R-1})}\right)\right) = L_R$

Thus (1.38) becomes $L_R < H_R(P) . D^{\left(\frac{R-1}{R}\right)} + \frac{R}{R-1}\left[1-D^{\left(\frac{R-1}{R}\right)}\right]$ for R >1 (1.10)

Cases II: when 0 < R < 1 Let n_i be the positive integer satisfying the inequality

$$-\log\left(\frac{p_i^R}{\sum p_i^R}\right) \le n_i < -\log\left(\frac{p_i^R}{\sum p_i^R}\right) + 1$$ Consider the interval

$$\delta_i = \left[-\log\left(\frac{p_i^R}{\sum p_i^R}\right), -\log\left(\frac{p_i^R}{\sum p_i^R}\right) + 1\right]$$ of length 1 .In every δ_i, there

lies one positive number n_i, such that

$$0 < -\log\left(\frac{p_i^R}{\sum p_i^R}\right) \le n_i < -\log\left(\frac{p_i^R}{\sum p_i^R}\right) + 1 \qquad (1.11)$$

It can be shown that the sequence $\{n_i\}, i = 1,2,3,......., N$ thus defined, satisfies (1.1).

Thus from (1.11), we get

$$n_i < -\log\left(\frac{p_i^R}{\sum p_i^R}\right) + 1$$

$$-\log D^{-n_i} < -\log\left[\left(\frac{p_i^R}{\sum p_i^R}\right) D^{-1}\right]$$

$$D^{-n_i} > \left[\frac{p_i^R}{\sum p_i^R}\right] D^{-1}$$

Raising above inequality by $\frac{R-1}{R}$, we get

$$D^{-n_i\left(\frac{R-1}{R}\right)} < \left(\frac{p_i^R}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)} \qquad (1.12)$$

Multiplying both sides of (1.12) by p_i , we get

$$P_i D^{-n_i\left(\frac{R-1}{R}\right)} < P_i \left(\frac{p_i^R}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow P_i D^{-n_i\left(\frac{R-1}{R}\right)} < P_i.(P_i)^{R-1} \left(\frac{1}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow P_i D^{-n_i\left(\frac{R-1}{R}\right)} < (P_i)^{1-1+R} \left(\frac{1}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow P_i D^{-n_i\left(\frac{R-1}{R}\right)} < .(P_i)^{R} \left(\frac{1}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)} \qquad (1.13)$$

Summing over $i = 1,2,3,\ldots\ldots.,N$ both sides, we get

$$\Rightarrow \sum P_i . D^{-n_i\left(\frac{R-1}{R}\right)} < (\sum P_i^{\,R})\left(\frac{1}{\sum p_i^R}\right)^{\frac{R-1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow \sum P_i . D^{-n_i\left(\frac{R-1}{R}\right)} < .(\sum P_i^{\,R})^{1-1+\frac{1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow \sum P_i . D^{-n_i\left(\frac{R-1}{R}\right)} < .(\sum P_i^{\,R})^{\frac{1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow -\sum P_i . D^{-n_i\left(\frac{R-1}{R}\right)} > -.(\sum P_i^{\,R})^{\frac{1}{R}} D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow 1-\sum P_i . D^{-n_i\left(\frac{R-1}{R}\right)} > 1-.(\sum P_i^{\,R})^{\frac{1}{R}} D^{\left(\frac{1-R}{R}\right)} \qquad (1.11)$$

We know $\frac{R}{R-1} < 0$ if $0 < R < 1$

Multiplying $\frac{R}{R-1}$ by both sides of (1.11), we get

$$\Rightarrow \frac{R}{R-1}\left[1-\left(\sum P_i . D^{-n_i(\frac{R}{R-1})}\right)\right] < \frac{R}{R-1}\left[1-\left(\sum P_i^{\,R}\right)^{\frac{1}{R}} . D^{\left(\frac{1-R}{R}\right)}\right]$$

$$\Rightarrow \frac{R}{R-1}\left[1-\left(\sum P_i . D^{-n_i(\frac{R}{R-1})}\right)\right] < \frac{R}{R-1} - \frac{R}{R-1}\left[\left(\sum P_i^{\,R}\right)^{\frac{1}{R}} . D^{\left(\frac{1-R}{R}\right)}\right]$$

$$\Rightarrow \frac{R}{R-1}\left[1-\left(\sum P_i . D^{-n_i(\frac{R}{R-1})}\right)\right] < \frac{R}{R-1} - \frac{R}{R-1}\left[\left(\sum P_i^{\,R}\right)^{\frac{1}{R}} . D^{\left(\frac{1-R}{R}\right)}\right]$$

$$\Rightarrow \frac{R}{R-1}\left[1-\left(\sum P_i . D^{-n_i(\frac{R}{R-1})}\right)\right] < \frac{R}{R-1} - \frac{R}{R-1}\left[1-1+\left(\sum P_i^{\,R}\right)^{\frac{1}{R}} .\right] . D^{\left(\frac{1-R}{R}\right)}$$

$$\Rightarrow \frac{R}{R-1}\left[1-\left(\sum P_i.D^{-n_i(\frac{R}{R-1})}\right)\right] < \frac{R}{R-1}\left[1-D^{\left(\frac{1-R}{R}\right)}\right] + \frac{R}{R-1}\left[1-\left\{\sum P_i^R\right\}^{\frac{1}{R}}.\right].D^{\left(\frac{1-R}{R}\right)} \quad (1.15)$$

But $$\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{N} P_i^R\right)^{\frac{1}{R}}\right) = H_R(P)$$

And $$\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{N} P_i D^{-n_i(\frac{R}{R-1})}\right)\right) = L_R$$

Thus (1.15) becomes

$$L_R < H_R(P).D^{\left(\frac{R-1}{R}\right)} + \frac{R}{R-1}\left[1-D^{\left(\frac{R-1}{R}\right)}\right] \quad \text{for } 0 < R < 1 \qquad (1.16)$$

Thus from (1.10) and (1.16), we get

$$L_R < H_R(P).D^{\left(\frac{R-1}{R}\right)} + \frac{R}{R-1}\left[1-D^{\left(\frac{R-1}{R}\right)}\right] \quad \text{for } R \in R^+ \qquad (1.17)$$

Theorem: For all integer D > 1

$$\sum P_i D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} = \left\{\sum P_i^R\right\}^{\frac{1}{R}} \qquad \text{for } R \in R^+ \qquad (1.18)$$

where $\bar{n}_i = -\log_D\left(p_i^R \Big/ \sum p_i^R\right)$

Proof: Since $$\bar{n}_i = -\log_D\left(p_i^R \Big/ \sum p_i^R\right) \qquad (1.19)$$

It can be written as

$$-\log_D D^{-\bar{n}_i} = -\log_D\left(p_i^R \Big/ \sum p_i^R\right)$$

$$D^{-\bar{n}_i} = \left(p_i^R \Big/ \sum p_i^R \right) \tag{1.50}$$

Raising power $\frac{R-1}{R}$ both sides of (1.50), we get

$$D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} = \left(p_i^R \Big/ \sum p_i^R \right)^{\left(\frac{R-1}{R}\right)}$$

$$D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} = \left(p_i^{R-1} \Big/ \left(\sum P_i^R\right)^{\left(\frac{R-1}{R}\right)} \right) \tag{1.51}$$

Multiplying both sides of (1.51) by P_i, we get

$$P_i D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} = = \left(p_i^R \Big/ \left(\sum P_i^R\right)^{\left(\frac{R-1}{R}\right)} \right) \tag{1.52}$$

Summing over $i = 1,2,3,\ldots\ldots\ldots,N$ both sides of (1.52), we get

$$\sum P_i D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} = \left(\sum P_i^R \Big/ \left(\sum P_i^R\right)^{\left(\frac{R-1}{R}\right)} \right)$$

$$= \left(\sum P_i^R\right)^{\frac{1}{R}} \tag{1.53}$$

Thus from (1.53), we get

$$\sum P_i D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} = \left(\sum P_i^R\right)^{\frac{1}{R}} \quad \text{for } R \in R^+ \tag{1.51}$$

Theorem: For every code length $\{n_i; i\} = 1,2,3,\ldots\ldots, N.$, L_R can be made to satisfy

$$L_R > H_R(P) + \frac{R}{R-1}\left(1 - D\right) \tag{1.55}$$

Proof: To prove this theorem we consider the following cases:

Cases I: when R >1

Suppose $\bar{n}_i = -\log\left(p_i^R \Big/ \sum p_i^R \right)$ (1.56)

Clearly $\bar{n}_i$ and $\bar{n}_i + 1$ satisfy the 'equality' in Holder's Inequality[12]. Moreover. $\bar{n}_i$ satisfies Kraft's Inequality (1.1)

Suppose n_i is the unique integer between $\bar{n}_i$ and $\bar{n}_i + 1$, and then obviously, n_i satisfies Kraft's Inequality (1.1), we have

$\bar{n}_i \le n_i < \bar{n}_i + 1$ Now consider $\bar{n}_i \le n_i$ It can be written as

$$-\log D^{-\bar{n}_i} \le -\log D^{-n_i}, \qquad D^{-in_i} \le D^{-\bar{n}_i} \tag{1.57}$$

Raising power $\frac{R}{R-1}$ both sides of (1.57), we get

$$D^{-n_i\left(\frac{R-1}{R}\right)} \le D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} \tag{1.58}$$

Multiply by P_i both sides of (1.58), we get

$$P_i D^{-n_i\left(\frac{R-1}{R}\right)} \le P_i D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} < D P_i D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} \tag{1.59}$$

Summing over $i = 1,2,3,\ldots\ldots\ldots,N$ both sides of (1.59), we get

$$\sum P_i D^{-n_i\left(\frac{R-1}{R}\right)} < D \sum P_i D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} \tag{1.60}$$

Using (1.51) in (1.60), we get

$$\sum P_i D^{-n_i\left(\frac{R-1}{R}\right)} < D\left(\sum P_i^R\right)^{\frac{1}{R}}$$

$$\Rightarrow -\sum P_i D^{-n_i\left(\frac{R-1}{R}\right)} > -\left(\sum P_i^R\right)^{\frac{1}{R}}.D$$

$$\Rightarrow 1-\sum P_i D^{-n_i\left(\frac{R-1}{R}\right)} > 1-\left(\sum P_i^R\right)^{\frac{1}{R}}.D \qquad (1.61)$$

We know $\frac{R}{R-1} > 0$ if $R > 1$

Multiplying both sides of (1.61) by $\frac{R}{R-1}$, we get

$$\frac{R}{R-1}\left(1-\left(\sum P_i D^{-n_i\left(\frac{R}{R-1}\right)}\right)\right) > \frac{R}{R-1}\left(1-\left(\sum P_i^R\right)^{\frac{1}{R}}.D\right)$$

$$\Rightarrow \frac{R}{R-1}\left(1-\left(\sum P_i D^{-n_i\left(\frac{R}{R-1}\right)}\right)\right) > \frac{R}{R-1}-\frac{R}{R-1}\left(\sum P_i^R\right)^{\frac{1}{R}}.D$$

$$\Rightarrow \frac{R}{R-1}\left(1-\left(\sum P_i D^{-n_i\left(\frac{R}{R-1}\right)}\right)\right) > \frac{R}{R-1}-\frac{R}{R-1}\left(1-1+\sum P_i^R\right)^{\frac{1}{R}}.D$$

$$\Rightarrow \frac{R}{R-1}\left(1-\left(\sum P_i D^{-n_i\left(\frac{R}{R-1}\right)}\right)\right) > \frac{R}{R-1}(1-D)+\frac{R}{R-1}\left(1-\sum P_i^R\right)^{\frac{1}{R}}.D \qquad (1.62)$$

But $$\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i^R\right)^{\frac{1}{R}}\right) = H_R(P)$$

And $\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i D^{-n_i\left(\frac{R}{R-1}\right)}\right)\right) = L_R$ Thus (1.63) becomes

$$L_R > H_R(P)+\frac{R}{R-1}(1-D) \quad \text{for } R > 1 \qquad (1.63)$$

Cases II: when 0 < R <1Suppose $\bar{n}_i = -\log\left(p_i^R \Big/ \sum p_i^R \right)$ (1.61)

Clearly $\bar{n}_i$ and $\bar{n}_i + 1$ satisfy the 'equality' in Holder's Inequality [12]. Moreover. $\bar{n}_i$ Satisfies Kraft's inequality (1.1). Suppose n_i is the unique integer between $\bar{n}_i$ and $\bar{n}_i + 1$, and then obviously, n_i satisfies (1.1), we have $\bar{n}_i \le n_i < \bar{n}_i + 1$ Now consider

$\bar{n}_i \le n_i$, It can be written as

$-\log D^{-\bar{n}_i} \le -\log D^{-n_i}$, $D^{-in_i} \le D^{-\bar{n}_i}$ (1.65)

Raising power $\frac{R}{R-1}$ both sides of , we get, $D^{-n_i\left(\frac{R-1}{R}\right)} \ge D^{-\bar{n}_i\left(\frac{R-1}{R}\right)}$

Multiplying both sides of (1.66) by P_i ,we get

$$P_i D^{-n_i\left(\frac{R-1}{R}\right)} \ge P_i D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} > D P_i D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} \qquad (1.66)$$

Summing over $i = 1,2,3,\ldots\ldots\ldots,N$ both sides of (1.66), we get

$$\sum P_i D^{-n_i\left(\frac{R-1}{R}\right)} > D \sum P_i D^{-\bar{n}_i\left(\frac{R-1}{R}\right)} \qquad (1.67)$$

Using (1.51) in (1.67), we get

$$\sum P_i D^{-n_i\left(\frac{R-1}{R}\right)} > D\left(\sum P_i^R\right)^{\frac{1}{R}}$$

$$\Rightarrow -\sum P_i D^{-n_i\left(\frac{R-1}{R}\right)} < -\left(\sum P_i^R\right)^{\frac{1}{R}}.D$$

$$\Rightarrow 1 - \sum P_i D^{-n_i\left(\frac{R-1}{R}\right)} < 1 - \left(\sum P_i^R\right)^{\frac{1}{R}}.D \qquad (1.68)$$

We know $\frac{R}{R-1} < 0$ if $0 < R < 1$

Multiplying both sides of (1.68) $\frac{R}{R-1}$, we get

$$\frac{R}{R-1}\left(1-\left(\sum P_i D^{-n_i(\frac{R}{R-1})}\right)\right) > \frac{R}{R-1}\left(1-\left(\sum P_i^{R}\right)^{\frac{1}{R}}.D\right)$$

$$\Rightarrow \frac{R}{R-1}\left(1-\left(\sum P_i D^{-n_i(\frac{R}{R-1})}\right)\right) > \frac{R}{R-1} - \frac{R}{R-1}\left(\sum P_i^{R}\right)^{\frac{1}{R}}.D$$

$$\Rightarrow \frac{R}{R-1}\left(1-\left(\sum P_i D^{-n_i(\frac{R}{R-1})}\right)\right) > \frac{R}{R-1} - \frac{R}{R-1}\left(1-1+\sum P_i^{R}\right)^{\frac{1}{R}}.D \qquad (1.69)$$

$$\Rightarrow \frac{R}{R-1}\left(1-\left(\sum P_i D^{-n_i(\frac{R}{R-1})}\right)\right) > \frac{R}{R-1}(1-D) + \frac{R}{R-1}\left(1-\sum P_i^{R}\right)^{\frac{1}{R}}.D$$

But $$\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i^{R}\right)^{\frac{1}{R}}\right) = H_R(P)$$

And $$\frac{R}{R-1}\left(1-\left(\sum_{i=1}^{n} P_i D^{-n_i(\frac{R}{R-1})}\right)\right) = L_R$$

Thus (1.69) becomes

$$L_R > H_R(P) + \frac{R}{R-1}(1-D) \quad \text{for } 0 < R < 1 \qquad (1.70)$$

Thus form (1.63) and (1.70), we get

$$L_R > H_R(P) + \frac{R}{R-1}(1-D) \quad \text{for } R \in R^{+} \qquad (1.71)$$

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