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ON ANALYSES OF PRIME GAMMA RINGS AND PRIME GAMMA RADICALS

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ABSTRACT

The prime ideals and prime radicals in the projective product of Gamma-rings are extensively studied in this paper. It is shown that the projective product of any two gamma rings can never be prime unless the two component gamma rings are factor gamma rings in which case the prime nature cannot be predicted.

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1.Introduction:

In study of Gamma-ring theory which was introduced by Nobusawa [7] and later re-defined by Barnes [10], different kinds of radicals play an important role. There is a very strong theory of various radicals on general rings and Banach algebras [1,6]. Many prominent mathematicians have extended fruitfully many significant technical results on radicals of general ring to the radicals of Gamma-ring [2,3,4,5,8,9,11].

2.Basic Terminologies:

The following terminologies are used in our main results as described below:

Definition 2.1: A gamma ring (X, Γ) in the sense of Nabusawa is said to be **simple** if for any two nonzero elements $x, y \in X$, there exist $\gamma \in \Gamma$ such that $x\gamma y \neq 0$.

Definition 2.2: If I is an additive subgroup of a gamma ring (X, Γ) and $X\Gamma I \subseteq I$ (or $I\Gamma X \subseteq$I), then I is called a left (or right) gamma ideal of X. If I is both left and right gamma ideal then it is said to be a gamma ideal of (X, Γ) or simply an ideal.

Definition 2.3: An ideal I of a gamma ring (X, Γ) is said to be **prime** if for any two ideals A and B of X, $A\Gamma B \subseteq I \Rightarrow A \subseteq I \text{ or } B \subseteq I$..

Definition 2.4: Let (X, Γ) be a gamma ring with left and right operator rings L and R respectively. X is said to have a **left (or right) unity** if there exist $d_1, d_2, \ldots\ldots, d_n \in X \text{ and } \delta_1, \delta_2, \ldots\ldots \delta_n \in \Gamma$ such that for all $x \in X$, $\sum_{i=1}^{n} d_i \delta_i x = x$ (or $\sum_{i=1}^{n} x\delta_i d_i = x$).

X is said to have a **strong left (or strong right) unity** if there exist $d \in X, \delta \in \Gamma$ such that

$d\delta x = x$ or $x\delta d = x$ for all $x \in X$.

An ideal I of X will be called **left modular (left strongly modular)** if the factor gamma ring X/I has a left unity (strong left unity). **Right modular and right strongly modular ideals** are similarly defined.

Definition 2.5: A gamma ring (X, Γ) is said to be a **prime gamma ring** if $x\Gamma X \Gamma x = 0, with\ x, y \in X$ implies either $x = 0 \text{ or } y = 0$.

Definition 2.6: An element a of a gamma ring (X, Γ) is strongly nilpotent if there exist a positive integer n such that $(a\Gamma)^n a = (a\Gamma a\Gamma a\Gamma \ \ldots\ldots a\Gamma)a = 0$. A subset S of X is strongly nil if each of its elements is strongly nilpotent. S is strongly nilpotent if there exist a positive integer n such that $(S\Gamma)^n S = 0$. Clearly a strongly nilpotent set is also strongly nil.

Definition 2.7: A subset S of X is an m-system in X if $S = \emptyset$ or if $a, b \in S$ implies

$< a > \Gamma < b > \cap S \neq \emptyset$. The prime radical of X which is denoted by ‘P(X), is defined as the set of elements x in X such that every m-system containing x contains 0. Barnes has characterized ‘P(X) as the intersection of all prime ideals of X.

Definition 2.8: For a gamma ring (X, Γ), the smallest ideal containing a is called the principal ideal generated by a and is denoted by $< a >$. We have $< a >= Za + a\Gamma X + X\Gamma a + X\Gamma a\Gamma X$, where $Za = \{na : n \text{ is an integer}\}$.

Definition 2.9: Let (X_1, Γ_1) and (X_2, Γ_2) be two gamma rings. Let $X = X_1 \times X_2$ and

$\Gamma = \Gamma_1 \times \Gamma_2$. Then defining addition and multiplication on X and Γ by,

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$,

$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$

and $(x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2) = (x_1\alpha_1 y_1, x_2\alpha_2 y_2)$

for every $(x_1, x_2), (y_1, y_2) \in X$ and $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \Gamma$,

(X, Γ) is a gamma ring. We call this gamma ring as the **Projective product of gamma rings**.

3.Main Results:

Theorem 3.1: If (X, Γ) be a gamma ring with the strong right unity d, then no proper ideal of X contains d and hence $d \notin \mathcal{P}(X)$

Proof: Since d is the strong right unity of X, so there exists a $\delta \in \Gamma$ such that

$$x\delta d = x \text{ for all } x \in X \qquad \text{..........(i)}$$

Let P be an ideal of X such that $d \in P$. Obviously, $P \subseteq X$.

Let $x \in X$ be any element.

Then, $x\delta d \in X\Gamma P \subseteq P$ [Since P is an ideal of X]

$=> x \in P$ [Using (i)]

Thus we get, $x \in X => x \in P$ i.e $X \subseteq P$. So we get, $P = X$.

Thus no proper ideal of X contains d.

Since a prime ideal is a proper ideal, so no prime ideal of X contains d and so $\mathcal{P}(X)$ being the intersection of all prime ideals of X does not contain d i.e $d \notin \mathcal{P}(X)$. Hence the result.

Theorem 3.2: For a non zero gamma ring (X, Γ) with the strong right unity d, if every non zero ideal contains d, then (X, Γ) is prime.

Proof: Since (X, Γ) is a gamma ring with the strong right unity d, so by the previous result, there does not exist any proper ideal containing d. But in (X, Γ), every nonzero ideal contains d. So the only ideals of X are 0 and X itself.

Thus we have the only options,

$0\Gamma 0 = 0$, $0\Gamma X = 0$, $X\Gamma 0 = 0$ and $X\Gamma X = 0$

The fourth option is absurd, because $X\Gamma X = 0 \Rightarrow X = 0$, which is not true.

And the other three options show that,

$A\Gamma B = 0 \Rightarrow A = 0$ *or* $B = 0$, where A *and* B are ideals of X.

Hence (X, Γ) is a prime gamma ring and hence the result.

Theorem 3.3: A non zero gamma ring (X, Γ) with the strong right unity d can never be nilpotent.

Proof: Since d is the strong right unity of X, so there exists a $\delta \in \Gamma$ such that

$$x\delta d = x \text{ for all } x \in X \qquad \text{..........(i)}$$

Since (X, Γ) is non zero, so d is also non zero. We show X is not nilpotent.

If possible, suppose, X is nilpotent. Then every element of X is nilpotent.

Since $d \in X$, so d is a nilpotent element.

So, for the above $\delta \in \Gamma$, there exists a positive integer n such that,

$(d\delta)^n d = 0$

$\Rightarrow d\delta d\delta \dots\dots . d\delta d = 0$

$\Rightarrow (d\delta d)\delta(d\delta d)\delta \dots\dots \delta(d\delta d) = 0$

$\Rightarrow d\delta d\delta \dots\dots \delta d = 0$ [Using (i)]

$\Rightarrow d = 0$ [By repeated application of (i)]

But $d = 0$ is a contradiction and so our supposition is wrong.

Hence X is not nilpotent and the result.

Theorem 3.4: Every prime gamma ring is simple.

Proof: Let (X, Γ) is a prime gamma ring. We show (X, Γ) is simple.

If possible, let, (X, Γ) be not simple.

Then there exists two non zero elements $x, y \in X$ such that, $x\gamma y = 0 \; \forall \gamma \in \Gamma$.

Let $A =< x >$ and $B =< y >$. Then $A \; and \; B$ are ideals of (X, Γ).

Let $a \in A\Gamma B$ be any element. Then $a \in< x > \Gamma < y >$

Since $x\gamma y = 0 \; \forall \gamma \in \Gamma$, so $a = 0$.

Thus we get, $A\Gamma B = 0$

Since (X, Γ) is a prime gamma ring, so $A\Gamma B = 0 => A = 0 \; or \; B = 0$

Without the loss of generality, let, $A = 0$

Then, $< x >= 0 => x = 0$, which contradicts that x is non zero.

Thus (X, Γ) is simple and hence the result.

Theorem 3.5: The projective product of any two gamma rings can never be prime unless the two component gamma rings are factor gamma rings in which case the prime-ness can not be predicted.

Proof: In my earlier works, it is being proved that the projective product of any two gamma rings can never be a simple gamma ring. So if the projective product of two gamma rings with the mentioned restriction is prime, then this product is also simple, as because by the previous result every prime gamma ring is simple as well.

For the restricted part, let (X, Γ) be the projective product of two gamma rings (X_1, Γ_1) and (X_2, Γ_2). If there exists a prime ideal $P = A \times B$ of X, with $A \subsetneq X_1$ and $B \subsetneq X_2$, then A and B are prime ideals of X_1 and X_2 respectively, which is shown in result(7).

Since, A, B and P are prime ideals of X_1, X_2 and X respectively, so the factor gamma rings ${X_1}/{A}$, ${X_2}/{B}$ and ${X}/{P}$ are prime gamma rings. Now, it can be easily verified that,

$$X/P = {(X_1 \times X_2)}/{(A \times B)} = \left({X_1}/{A}\right) \times \left({X_2}/{B}\right)$$

That is, ${X}/{P}$ is the projective product of two gamma rings ${X_1}/{A}$ and ${X_2}/{B}$ and which is prime as well. Thus it is not possible for the projective product of two gamma rings with the mentioned restriction to be prime and hence the result.

Theorem 3.6: Let (X, Γ) be the projective product of two gamma rings (X_1, Γ_1) and (X_2, Γ_2). Then any two ideals of X_1 and X_2 give rise to an ideal of X and vice versa.

Proof: Let A and B be two ideals of X_1 and X_2 respectively.

Then $X_1 \Gamma_1 A \subseteq A,\ A \Gamma_1 X_1 \subseteq A$ and $X_2 \Gamma_2 B \subseteq B,\ B \Gamma_2 X_2 \subseteq B$

Since $A \subseteq X_1$ *and* $B \subseteq X_2$, so, $C = A \times B \subseteq X_1 \times X_2 = X$.

Now, $X \Gamma C = (X_1 \times X_2)(\Gamma_1 \times \Gamma_2)(A \times B) = X_1 \Gamma_1 A \times X_2 \Gamma_2 B \subseteq A \times B = C$

$\Rightarrow X \Gamma C \subseteq C$. Similarly, $C \Gamma X \subseteq C$.

So, $C = A \times B$ is an ideal of X.

Conversely, let C be an ideal of X. Then C is of the form $A \times B$, where $A \subseteq X_1$ and $B \subseteq X_2$.

We show, A *and* B are ideals of X_1 and X_2 respectively.

Since C is an ideal of X, so, $X \Gamma C \subseteq C$ and $C \Gamma X \subseteq C$.

Now, $X \Gamma C \subseteq C \Rightarrow (X_1 \times X_2)(\Gamma_1 \times \Gamma_2)(A \times B) \subseteq A \times B \Rightarrow X_1 \Gamma_1 A \times X_2 \Gamma_2 B \subseteq A \times B$

$\Rightarrow X_1 \Gamma_1 A \subseteq A$ and $X_2 \Gamma_2 B \subseteq B$ ....(i)

Again, $C \Gamma X \subseteq C \Rightarrow A \Gamma_1 X_1 \subseteq A$ and $B \Gamma_2 X_2 \subseteq B$ ....(ii)

Thus from (i) and (ii), we get, $A \, and \, B$ are ideals of X_1 and X_2 respectively. Hence the result.

Theorem 3.7: Let (X, Γ) be the projective product of two gamma rings (X_1, Γ_1) and (X_2, Γ_2). Then every prime ideal of X gives rise to at least one prime ideal of X_1 or X_2 and conversely every prime ideal of X_1 or X_2 give rise to a prime ideal of X.

Proof: Let $P = A \times B$ be a prime ideal of X. Then P is a proper subset of X, so at least one of $A \, and \, B$ is a proper subset of X_1 and X_2 respectively.

Without the loss of generality, let, $A \subsetneq X_1$. We show, A is a prime ideal of X_1.

Let, I and J be two ideals of X_1 such that $I \Gamma_1 J \subseteq A => I \Gamma_1 J \times B \subseteq A \times B$

$=> I \Gamma_1 J \times B \Gamma_2 B \subseteq I \Gamma_1 J \times B \subseteq A \times B$ [Since B is an ideal, so $B \Gamma_2 B \subseteq B$]

$=> (I \times B)(\Gamma_1 \times \Gamma_2)(J \times B) \subseteq A \times B => I' \Gamma J' \subseteq A \times B$, where $I' = I \times B$ and $J' = J \times B$ are ideals of X.

Since $A \times B$ is a prime ideal so, $I' \Gamma J' \subseteq A \times B => I' \subseteq A \times B$ or $J' \subseteq A \times B$

If $I' \subseteq A \times B$ then $I \times B \subseteq A \times B => I \subseteq A$

And if $J' \subseteq A \times B$ then $J \times B \subseteq A \times B => J \subseteq A$

Thus we get, $I \Gamma_1 J \subseteq A => I \subseteq A$ or $J \subseteq A$

So, A is a prime ideal of X_1.

Converse part: Let A and B be two prime ideals of X_1 and X_2 respectively. Then, $M = A \times X_2$ and $N = X_1 \times B$ are two ideals of X. We show $M \, and \, N$ are prime ideals X.

For this, let, $P = P_1 \times P_2$ and $Q = Q_1 \times Q_2$ be two ideals of X such that, $P \Gamma Q \subseteq M$.

Then P_1, Q_1 and P_2, Q_2 are ideals of X_1 and X_2 respectively.

$=> (P_1 \times P_2)(\Gamma_1 \times \Gamma_2)(Q_1 \times Q_2) \subseteq M => P_1 \Gamma_1 Q_1 \times P_2 \Gamma_2 Q_2 \subseteq A \times X_2$

$=> P_1 \Gamma_1 Q_1 \subseteq A$ and $P_2 \Gamma_2 Q_2 \subseteq X_2$

Since A is a prime ideal of X_1, so, $P_1 \Gamma_1 Q_1 \subseteq A => P_1 \subseteq A \, or \, Q_1 \subseteq A$

If $P_1 \subseteq A$ then $P_1 \times P_2 \subseteq A \times X_2 => P \subseteq M$

Similarly if $Q_1 \subseteq A$ then $Q_1 \times Q_2 \subseteq A \times X_2 => Q \subseteq M$

Thus M is a prime ideal of X. Similarly, N is also a prime ideal of X. Hence the result.

Theorem 3.8: If (X, Γ) be the projective product of two gamma rings (X_1, Γ_1) and (X_2, Γ_2), then $\mathcal{P}(X_1) \times \mathcal{P}(X_2) \subseteq \mathcal{P}(X)$.

Proof: We know, $\mathcal{P}(X) =$ Intersection of all prime ideals of X

i.e $\mathcal{P}(X) = \cap P$, where P represents all prime ideals of X

Since every prime ideal of X gives rise to at least one prime ideal of X_1 or X_2, so,

$\mathcal{P}(X) = \cap A \times \cap B$, where some A *and* B are prime ideals of X_1 and X_2 respectively and some are not. Those A *and* B which are not prime are equal to X_1 and X_2 respectively.

$= \cap A \times \cap B$, where A *and* B are some prime ideals of X_1 and X_2 respectively.

[Since, $\cap A$, where A is either prime in X_1 or $A = X_1$ is equal to $\cap A$, where A is only prime X_1 and $\cap B$, where B is either prime in X_2 or $B = X_2$ is equal to $\cap B$, where A is only prime inX_2]

$\supseteq \cap A \times \cap B$, where A *and* B represents all prime ideals of X_1 and X_2 respectively.

$= \mathcal{P}(X_1) \times \mathcal{P}(X_2)$

Hence, $\mathcal{P}(X_1) \times \mathcal{P}(X_2) \subseteq \mathcal{P}(X)$.

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