THE EFFECT OF FOURIER TRANSFORM ON FUNCTIONS IN HILBERT SPACE

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ABSTRACT

The effect of Fourier transform on functions in Hilbert space was clearly stated .The smaller the variance of a quantity such as position or momentum, the more accurate will be its measurement.

Thus ,the Heisenberg inequality implies precisely ideas that the more accurately we are able to measure the momentum p,the less accurate will be any measurement of its position x,and vice versa.

Keywords: Fourier Transform: Heisenberg inequality: Hilbert Space: Fourier Series : Uncertainty Principle .

1. Introduction

The Fourier transform is, like Fourier series, completely compatible with the calculus of generalized functions. We have already noted that the Fourier transform, when defined, is a linear map, taking functions $f(x)$ on physical space to functions $\hat{f}(k)$ on frequency space. A critical question is precisely which function space should the theory be applied to. Not every function admits a Fourier transform in the classical sense† — the Fourier integral is required to converge, and this places restrictions on the function and its asymptotics at large distances.

It turns out the proper setting for the rigorous theory is the Hilbert space of complex valued square-integrable functions — the same infinite-dimensional vector space that lies at the heart of modern quantum mechanics.

The Hilbert space $L^2 = L^2\ (\mathcal{R})$ is the infinite-dimensional vector space consisting of all complex-valued functions $f(x)$ which are defined for all $x \in \mathcal{R}$ and have finite L^2 norm:

$$\|f\| = \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \qquad (1)$$

Hilbert space contains many more functions, and the precise definitions and identification of its elements
is quite subtle. The Hermitical inner product on the complex Hilbert space L^2 is prescribed in the usual manner,

$$\langle f; g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\ dx, \qquad (2)$$

so that $\|f\|^2 = \langle f; f \rangle$.The Cauchy –Schwarz inequality

$$|\langle f; g \rangle| \leq \|f\|\|g\| \qquad (3)$$

ensures that the inner product integral is finite whenever $f, g \in L^2$.

The goal of this paper is to see effect of Fourier transform between position and momentum, or multiplication and differentiation, related to uncertainty principle. And also, to explain the mean value of any $f(x)$ the position variable is given by its integral against the system's probability density.

2. Some properties of Fourier Transform

If $f(x) \in L^2$ is square-integrable, then its Fourier transform $\hat{f}(x) \in L^2$ is a well-defined, square-integrable function of the frequency variable k.

If $f(x)$ is continuously differentiable at a point x, then its inverse Fourier transform equals its value $f(x)$. More generally, if the right and left hand limits $f(x^-)$, $f(x^+)$, and also $f'(x^-), f(x^+)$ exist, then the inverse Fourier transform integral converges to the average value $\frac{1}{2}[f(x^-)+f(x^+)]$.

Thus, the Fourier transform $\hat{f} = \mathcal{F}[\, f \,]$ defines a linear transformation from L^2 functions of x to L^2 functions of k. In fact, the Fourier transform preserves inner products. This important result is known as Parseval's formula.

Theorem 2.1.If $\hat{f}\,(k) = \mathcal{F}[f(x)]$ and $\hat{g}\,(k) = \mathcal{F}[g(x)]$,then $\langle f; g\rangle = \langle \hat{f}, \hat{g}\rangle$,i.e.,

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \int_{-\infty}^{\infty} \hat{f}(k)\, \overline{\hat{g}(k)}\, dk \tag{4}$$

Proof :Let us sketch a formal proof that serves to motivate why this result valid.

By using the definition of the Fourier Transform to evaluate

$$\int_{-\infty}^{\infty} \hat{f}(k)\, \overline{\hat{g}(k)}\, dk = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{g(y)}\, e^{+ikx}\, dx\right) dk$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, \overline{g(y)} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{g(y)}\, e^{-ik\,(x-y)} dk\right) dxdy$$

Accordingly, the inner k integral can be replaced by a delta function $\delta(x-y)$,and hence $\int_{-\infty}^{\infty} \hat{f}(k)\, \overline{\hat{g}(k)}\, dk = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, \overline{g(y)}\, \delta(x-y) dxdy = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx.$

This completes the proof .

In particular, orthogonal functions, satisfying $\langle f, g\rangle = 0$, will have orthogonal Fourier transforms, $\langle \hat{f}, \hat{g}\rangle = 0$. Choosing $f = g$ in Parseval's formula (4) results in the Plancherel formula

$\|f\|^2 = \|\hat{f}\|^2$, or,explicitly, $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk$ (5)

Therefore, the Fourier transform $\mathcal{F}: L^2 \to L^2$ defines a unitary or norm-preserving transformation on Hilbert spaces, mapping L^2 functions of the physical variable x to L^2 functions of the frequency variable .

3. Quantum Mechanics and the Uncertainty Principle

The Heisenberg Uncertainty Principle states that in a physical system, certain quantities cannot be simultaneously measured with complete accuracy. For instance, the more precisely one measures the position of a particle, the less accuracy there will be in the measurement of its momentum; vice versa, the greater the accuracy in the momentum, the less certainty in its position. A similar uncertainty couples energy and time. Experimental verification of the uncertainty principle can be found even in fairly simple situations. Consider a light beam passing through a small hole.

The position of the photons is constrained by the hole; the effect of their momenta is in the pattern of light diffused on a screen placed beyond the hole. The smaller the hole, the more constrained the position, and the wider the image on the screen, meaning the less certainty there is in the observed momentum. This is not the place to discuss the philosophical and experimental consequences of Heisenberg's principle. What we will show is that the Uncertainty Principle is, in fact, a mathematical property of the Fourier transform!

In quantum theory, each of the paired quantities, e.g., position and momentum, are interrelated by the Fourier transform.. This Fourier transform-based duality between position and momentum, or multiplication and differentiation, lies at the heart of the Uncertainty Principle .In quantum mechanics, the wave functions of a quantum system are characterized as the elements of unit norm, $\|\varphi\| = 1$, belonging to the underlying state space, which, in a one-dimensional model of a single particle, is the Hilbert space $L^2 = L^2$ $(\mathbb{R})$ consisting of square integrable, complex valued functions of x. The squared modulus of the wave function, $|\varphi(x)|^2$, represents the probability density of the particle being found at position x. Consequently, the mean value of any function $f(x)$ of the position variable is given by its integral against the system's probability density, and denoted by.

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)\,|\varphi(x)|^2 dx. \qquad (6)$$

In particular,

$$\langle x\rangle = \int_{-\infty}^{\infty} x\,|\varphi(x)|^2 dx \qquad (7)$$

is the mean or average measured position of the particle, while Δx, defined by

$$(\Delta x)^2 = \langle (x-\langle x\rangle)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 \qquad (8)$$

is the variance ,that is ,the statistical deviation of the particle's measured position from the mean .We note that $\langle x^2\rangle = \int_{-\infty}^{\infty} x^2\,|\varphi(x)|^2 dx = \|x\varphi(x)\|^2$. (9)

On the other hand, the momentum variable p is related to the Fourier transform frequency via the de Broglie relation $p = \hbar k$, where

$$\hbar = \frac{h}{2\pi} \approx 1.055x10^{-34} \text{ Joule seconds} \qquad (10)$$

is Planck's constant, whose value governs the quantization of physical quantities. Therefore ,the mean value of any function of momentum $g(p)$ is given by its integral against the modulus of the Fourier transformed wave function:

$$\langle g(p)\rangle = \int_{-\infty}^{\infty} g(\hbar k)\,|\hat{\varphi}(k)|^2 dk.$$

(11)

In particular ,the mean of the momentum measurements of the particle is given by

$$\langle p\rangle = \hbar\int_{-\infty}^{\infty} k|\hat{\varphi}(k)|^2 dk = -i\,\hbar\int_{-\infty}^{\infty} \varphi'(x)\,\overline{\varphi(x)}dx = -i\hbar\langle\varphi';\varphi\rangle,$$

(12)

Where we used Parseval's formula formula (4) to convert to an integral over position, and by using the fact that the Fourier Transform of the derivative $f'(x)$ of the function is obtained by multiplication of its Fourier transform by ik:

$$\mathcal{F}[f'(x)] = ik\hat{f}(k) \qquad (13)$$

Based on [13],we can infer that $k\hat{\varphi}(k)$ is the Fourier transform of $-i\varphi'(x)$. Similarly,

$$(\Delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle = \langle p^2 \rangle - \langle p^2 \rangle$$

(14) is the squared variance in the momentum ,where, by a similar computation,

$$\langle p^2 \rangle = \hbar^2 \int_{-\infty}^{\infty} k^2 |\hat{\varphi}(k)|^2 dk = -\hbar^2 \int_{-\infty}^{\infty} \varphi(x)\overline{\varphi''(x)} dx = \hbar^2 \int_{-\infty}^{\infty} |\varphi'(x)|^2 dx = \hbar^2 \|\varphi'(x)\|^2. \quad (15)$$

With this interpretation, the uncertainty principle can be stated as follows.

Theorem 3.1 If $\varphi(x)$ is a wave function, so $\varphi(x) = 1$, then the variances in position and momentum satisfy the inequality

$$\Delta x \Delta p \geq \frac{1}{2}\hbar. \quad (16)$$

The smaller the variance of a quantity such as position or momentum ,the more accurate will be its measurement .Thus ,the more accurate will be its measurement.Thus,the Heisenberg inequality (16) quantifies the statement that the more accurately we are able to measure the momentum p, the less accurate will be any measurement of its position x,and vice versa.

Proof: For any value of the real parameter t,

$$0 \leq \|tx\varphi(x) + \varphi'(x)\|^2 = t^2\|x\varphi(x)\|^2 + t[\langle \varphi'(x); x\varphi(x)\rangle + \langle x\varphi(x); \varphi'(x)\rangle] + \|\varphi'(x)\|^2$$

(17)

The middle term can be evaluated as follows:

$$\langle \varphi'(x); x\varphi(x)\rangle + \langle x\varphi(x); \varphi'(x)\rangle = \int_{-\infty}^{\infty} \left[x\varphi'(x)\overline{\varphi(x)} + x\varphi(x)\overline{\varphi'(x)}\right] dx$$

$$= \int_{-\infty}^{\infty} x \frac{d}{dx} |\varphi(x)|^2 \, dx = \int_{-\infty}^{\infty} |\varphi(x)|^2 \, dx = -1,$$ where we employed integration by parts, noting that the boundary terms vanish since $\varphi(x) \to 0 \; as \; |x| \to \infty$. Thus ,as indicated in (9,15 & 17) reads

$$\langle x^2 \rangle t^2 - t + \frac{\langle p^2 \rangle}{\hbar^2} \geq 0 \text{ for all real } t.$$

The minimum value of the left hand side of this inequality occurs at $t_* = \frac{1}{(2\langle x^2\rangle)}$,where its value is

$\frac{\langle p^2\rangle}{\hbar^2} - \frac{1}{4\langle x^2\rangle} \geq 0$ and hence $\langle x^2\rangle\langle p^2\rangle \geq \frac{1}{4}\hbar^2$.

To obtain the Uncertainty Relation, one performs same calculation, but with $x - \langle x\rangle$ replacing x and $p - \langle p\rangle$ replacing p .

The result is

$$\langle(x - \langle x\rangle)^2\rangle t^2 - t + \frac{\langle(p-\langle p\rangle)^2\rangle}{\hbar^2} = (\Delta\, x)^2 t^2 - t + \frac{(\Delta p)^2}{\hbar^2} \geq 0. \qquad (18)$$

Substituting $t = {}^{1}/_{(2(\Delta\, x)^2)}$ gives $\Delta x \Delta p \geq \frac{1}{2}\hbar$.*which is the Heisenberg inequality.*

4. Conclusions

This paper is based on the use of Fourier Transform for determining Heisenberg inequality . The relationship between measurement of the position of a particle and accuracy of the measurement of the momentum, were inter related in such way that in reverse way. The linear equation which gives the *Heisenberg inequality explained the inverse relation to between measurement of position and accuracy measurement momentum.*

The method of Fourier Transform is efficient and accurate in relation to evaluate Hilbert spaces and Heisenberg inequality .Given theorem which are given in $(4) and$ (16) *supports the effect of Fourier Transform on Hilbert Spaces.*

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