FUZZY β-IRRESOLUTE MAPPING

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ABSTRACT

Some properties of fuzzy β-irresolute mapping (formerly known as fuzzy $M\beta$-continuous mapping [8]) have been studied here. Also it has been shown that fuzzy irresolute mapping [11] and fuzzy β-irresolute mapping are independent notions. In the last section some applications of fuzzy β-irresolute mapping have been discussed.

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INTRODUCTION

Throughout the paper, by (X, τ) or simply by X we mean a fuzzy topological space (fts, for short) in the sense of Chang [4]. A fuzzy set [16] A is a mapping from a nonempty set X into a closed interval $I = [0, 1]$. The support [13] of a fuzzy set A in X will be denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [13] with the singleton support $x \in X$ and the value $t(0 < t \leq 1)$ at x will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 in X respectively. The complement [16] of a fuzzy set A in X will be denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for all $x \in X$. For two fuzzy sets A and B in X, we write $A \leq B$ if and only if $A(x) \leq B(x)$, for each $x \in X$, and AqB means A is quasi-coincident (q-coincident, for short) with B [13] if $A(x) + B(x) > 1$, for some $x \in X$. The negation of these two statements will be denoted by $A \not\leq B$ and $A\bar{q}B$ respectively. clA and $intA$ of a fuzzy set A in X respectively stand for the fuzzy closure [4] and fuzzy interior [4] of A in X. A fuzzy set A in X will be called fuzzy semiopen [2] (resp., fuzzy β-open [1], fuzzy preopen [12]) if $A \leq cl\,intA$ (resp., $A \leq$

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$cl\ int\ clA, A\ \leq int\ clA$). The set of all fuzzy semiopen (resp., fuzzy β-open) sets of X will be denoted by $SO(X)$ (resp., $\beta O(X)$). The complement of a fuzzy semiopen (resp., fuzzy β-open, fuzzy preopen) set A in X is called fuzzy semiclosed [2] (resp., fuzzy β-closed [1], fuzzy preclosed 12]). The smallest fuzzy semiclosed (resp., fuzzy β-closed, fuzzy preclosed) set containing a fuzzy set A in X is called fuzzy semiclosure [2] (resp., fuzzy β-closure [1], fuzzy preclosure [12]) of A and is denoted by $sclA$ (resp., βclA, $pclA$). A fuzzy set B in X is said to be a β-nbd [1] of a fuzzy point x_t in X if there exists a fuzzy β-open set U in X such that $x_t\ \leq U\ \leq B$. A fuzzy set B is called a fuzzy β-q-nbd [8] of a fuzzy point x_t in X if there is a fuzzy β-open set U in X such that $x_t qU\ \leq B$.

1. FUZZY $\boldsymbol{\beta}$-IRRESOLUTE MAPPING : SOME CHARACTERIZATIONS

In this section fuzzy β-irresolute mapping has been characterized in different ways.

DEFINITION 1.1. A fuzzy mapping $f : X\ \to Y$ is said to be fuzzy β-irresolute (fuzzy $M\beta$-continuous mapping [8]) if $f^{-1}(A)$ is fuzzy β-open in X for each fuzzy β-open set A in Y.

THEOREM 1.2. Let $f : X\ \to Y$ be a fuzzy function. Then the following are equivalent :

(a) f is fuzzy β-irresolute,
(b) for each fuzzy point x_t in X and each fuzzy β-open set A in Y such that $f(x_t)\ \leq A$, there exists a fuzzy β-open set B in X such that $x_t\ \leq B$ and $f(B)\ \leq A$,
(c) $f^{-1}(B)$ is fuzzy β-closed in X for each fuzzy β-closed set B in Y,
(d) for each fuzzy point x_t in X, the inverse of each fuzzy β-nbd B of $f(x_t)$ in Y is a fuzzy β-nbd of x_t in X,
(e) for each fuzzy point x_t in X and each fuzzy β-nbd B of $f(x_t)$, there exists a fuzzy β-nbd C of x_t in X such that $f(C)\ \leq\ B$,
(f) for each fuzzy set D in X, $f(\beta clD)\ \leq\ \beta cl\, f(D)$,
(g) for each fuzzy set B in Y, $\beta cl\left(f^{-1}(B)\right)\ \leq\ f^{-1}(\beta cl\, B)$.

PROOF. (b) $\Rightarrow$ (a). Let A be a fuzzy β-open set in Y and x_t, fuzzy point in $f^{-1}(A)$. Then$x_t\ \leq\ f^{-1}(A)$, i.e., $f(x_t)\ \leq A$. By (b), there exists a fuzzy β-open set B in X such that $x_t\ \leq B$ and $f(B) \leq A$. Thus $B\ \leq\ f^{-1}(A)$. We have to show that $f^{-1}(A)\ \leq cl\ int\ clf^{-1}(A)$. As $B\ \in\ \beta O(X)$, $x_t\ \leq B\ \leq cl\ int\ clB\ \leq cl\ int\ clf^{-1}(A)$. As $x_t\ \leq\ f^{-1}(A)$, $f^{-1}(A)\ \leq cl\ int\ clf^{-1}(A)$.

(a) $\Rightarrow$ (c). Let Bbe any fuzzy β-closed set in Y. Then $1_Y \setminus B \in \beta O(Y)$. By (a), $f^{-1}(1_Y \setminus B=1X \setminus f-1B \in \beta O(X)$ and so $f-1(B)$ is fuzzy β-closed in X.

(c) $\Rightarrow$ (a). Straightforward.

(a) $\Rightarrow$ (d). Let x_t be a fuzzy point in X and B, a fuzzy β-nbd of $f(x_t)$ in Y. Then there exists $U \in \beta O(Y)$ such that $f(x_t) \leq U \leq B$. Then $x_t \leq f^{-1}(U) \leq f^{-1}(B)$. Since $U \in \beta O(Y)$, by (a) $f^{-1}(U) \in \beta O(X)$ and hence the result.

(d) $\Rightarrow$ (e). Since $ff^{-1}(B) \leq B$, the result follows by taking $C = f^{-1}(B)$.

(e) $\Rightarrow$ (b). Let x_t be a fuzzy point in X and$A$, any fuzzy β-open set in Y such that $f(x_t) \leq A$. Then A is fuzzy β-nbd of $f(x_t)$ in Y. By (e), there exists a fuzzy β-nbd C of x_t in X such that $f(C) \leq A$. Therefore, there exists $U \in \beta O(X)$ such that $x_t \leq U \leq C$ and so $f(U) \leq f(C) \leq A \Rightarrow f(U) \leq A$.

(c) $\Rightarrow$ (f). Let D be any fuzzy set in X. Then $\beta cl\, f(D)$ is fuzzy β-closedin Y. By (c), $f^{-1}(\beta cl\, f(D))$ is fuzzy β-closed in X. Now $D \leq f^{-1}f(D) \leq f^{-1}(\beta cl\, f(D))$, i.e., $\beta cl\, D \leq \beta cl\, f^{-1}\left(\beta cl\, f(D)\right) = f^{-1}(\beta cl\, f(D))$. Therefore, $f(\beta cl\, D) \leq \beta cl\, f(D)$.

(f) $\Rightarrow$ (c). Let B be any fuzzy β-closed set in Y. Put $D = f^{-1}(B)$. By (f), $f(\beta cl\, D) \leq \beta cl\, f(D) = \beta cl\left(f\left(f^{-1}(B)\right)\right) \leq \beta cl\, B = B$. Thus $\beta cl\, D \leq f^{-1}\left(f(\beta cl\, D)\right) \leq f^{-1}(B) = D$. Hence $D = f^{-1}(B)$ is fuzzy β-closed in X.

(f) $\Rightarrow$ (g). Let $B \in I^Y$. Again let $D = f^{-1}(B)$. By (f), $f(\beta cl\, D) \leq \beta cl\, f(D)$, i.e., $\beta cl\, D \leq f^{-1}(\beta cl\, f(D))$, i.e., $\beta cl\, f^{-1}(B) \leq f^{-1}\left(\beta cl\, f\left(f^{-1}(B)\right)\right) \leq f^{-1}(\beta cl\, B)$.

(g) $\Rightarrow$ (f). Let $D \in I^X$. By (g), $\beta cl\left(f^{-1}f(D)\right) \leq f^{-1}\left(\beta cl\, f(D)\right) \Rightarrow \beta cl\, D \leq f^{-1}\left(\beta cl\, f(D)\right) \Rightarrow f(\beta cl\, D) \leq \beta cl\, f(D)$.

THEOREM 1.3. A mapping $f : X \to Y$ is fuzzy β-irresolute iff for each fuzzy point x_t in X and any fuzzy β-open β-q-nbd V of $f(x_t)$ in Y, there exists a fuzzy β-open β-q-nbd U of x_t in X such that $f(U) \leq V$.

PROOF. Let $f : X \to Y$ be fuzzy β-irresolute and $x_t$be a fuzzy point in X. Let V be a fuzzy β-open β-q-nbd of $f(x_t)$ in Y. Then $f^{-1}(V)$ ($= U$, say) is a fuzzy β-open β-q-nbd of $f(x_t)$ in X such that $f(U) \leq V$.

Conversely, let x_t be any fuzzy point in X and V be any fuzzy β-open set containing $f(x_t)$. Let m_t be a positive integer such that $1/m_t < t$. Then $0 < 1 - t + 1/n = \beta_n$ (say) < 1, for all $n \geq m_t$. Now $y_{\beta_n} qV$ for each $n \geq m_t$, where $y = f(x)$. Then by hypothesis, there exists a fuzzy β-open set U_n in X such that $x_{\beta_n} qU_n$ and $f(U_n) \leq V$, for all $n \geq m_t$. Put $U = \bigcup_{n \geq m_t} U_n$. Then $U \in \beta O(X)$ such that $f(U) \leq V$. Also $\beta_n + U_n(x) > 1$, for all $n \geq m_t \Rightarrow 1 - t + 1/n + U_n(x) > 1$, for all $n \geq m_t \Rightarrow t < U_n(x) + 1/n$, for all $n \geq m_t \Rightarrow t \leq \sup\{U_n(x): n \geq m_t\} = U(x) \implies x_t \leq U$. Hence by Theorem 1.2, f is fuzzy β-irresolute.

2. FUZZY IRRESOLUTE AND FUZZY β-IRRESOLUTE MAPPING

In this section it has been shown that fuzzy irresolute mapping [11] and fuzzy β-irresolute mapping are independent notions.

First we recall the definition from [11] for ready reference.

DEFINITION 2.1. A fuzzy mapping $f : X \to Y$ is said to be fuzzy irresolute if $f^{-1}(A)$ is fuzzy semiopen in X for each fuzzy semiopen set A in Y.

REMARK 2.2. It is clear from the following two examples that fuzzy irresolute mapping and fuzzy β-irresolute mapping are independent notions.

EXAMPLE 2.3. Let $X = \{a, b\}, \tau = \{0_X, 1_X, A\}, \tau_1 = \{0_X, 1_X, C\}$ where $A(a) = 0.5, A(b) = 0.4, C(a) = 0.6, C(b) = 0.5$. Then (X, τ) and (X, τ_1) are fts's. Consider the fuzzy mapping $f : (X, \tau) \to (X, \tau_1)$ defined by $f(a) = b,\ f(b) = a$. We claim that f is fuzzy β-irresolute but not fuzzy irresolute mapping. The collection of all fuzzy semiopen sets in (X, τ) is $\{0_X, 1_X, A, U\}$ where $A \leq U \leq 1_X \setminus A$ and that of in (X, τ_1) is $\{0_X, 1_X, C, V\}$ where $V \geq C$. Again any fuzzy set in (X, τ) is fuzzy β-open in (X, τ) and the collection of all fuzzy β-open sets in (X, τ_1) is $\{0_X, 1_X, C, W\}$ where $W \nleq 1_X \setminus C$.

Let B be a fuzzy semiopen set in (X, τ_1) defined by $B(a) = B(b) = 0.6$. Now $[f^{-1}(B)](a) = B\, f(a) = B(b) = 0.6$, $[f^{-1}(B)](b) = B\, f(b) = B(a) = 0.6$, and so $f^{-1}(W) \notin SO(X, \tau)$. Therefore, f is not fuzzy irresolute mapping. Since any fuzzy set in (X, τ) is fuzzy β-open in (X, τ), f is fuzzy β-irresolute.

EXAMPLE 2.4. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, B\}$ where $A(a) = 0.4, A(b) = 0.7, B(a) = 0.6, B(b) = 0.7$. Then (X, τ) and (X, τ_1) are fts's. Now fuzzy semiopen sets in (X, τ) are $0_X, 1_X, A, V$ where $V \geq A$ and that of fuzzy β-open sets in (X, τ) are $0_X, 1_X, A, U$ where $U \nleq 1_X \setminus A$. Again fuzzy semiopen sets in (X, τ_1) are $0_X, 1_X, B, C$ where $C \geq B$ and that of fuzzy β-open sets in (X, τ_1) are $0_X, 1_X, B, W$ where $W \nleq 1_X \setminus B$. Consider the fuzzy identity mapping $i : (X, \tau) \to (X, \tau_1)$. We claim that i is fuzzy irresolute but not fuzzy β-irresolute mapping. Infact, $[i^{-1}(C)](a) = C(i(a)) = C(a) \geq B(a)$ and $[i^{-1}(C)](b) = C(i(b)) = C(b) \geq B(b)$ and $B \geq A \Rightarrow i^{-1}(C) \geq A$ which shows that i is fuzzy irresolute. But $W(a) = 0.6, W(b) = 0.3$ being a fuzzy β-open set in (X, τ_1) and $i^{-1}(W) = W \notin \beta O(X, \tau)$ and so i is not fuzzy β-irresolute mapping.

3. APPLICATIONS

Let us recall some definitions for ready references.

DEFINITION 3.1 [4]. Let A be a fuzzy set in an fts X. A collection $\mathcal{U}$ of fuzzy sets in X is called a fuzzy cover of A if $\sup\{U(x): U \in \mathcal{U}\} = 1$, for each $x \in suppA$. In particular, if $A = 1_X$, we get the definition of fuzzy cover of the fts X.

DEFINITION 3.2 [6]. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in an fts X is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\cup\, \mathcal{U}_0 \geq A$. In particular, if $A = 1_X$, then the requirement on $\mathcal{U}_0$ is $\cup\, \mathcal{U}_0 = 1_X$.

DEFINITION 3.3 [9]. An fts X is said to be a fuzzy semicompact space if every cover of X by fuzzy semiopen sets has a finite subcover.

DEFINITION 3.4. An fts X is said to be fuzzy S-closed [10] (resp., fuzzy s-closed [15]) if every fuzzy cover $\mathcal{U}$ of X by fuzzy semiopen sets in X has a finite subfamily $\mathcal{U}_0$ such that $\bigcup_{U \in \mathcal{U}_0} clU = 1_X$ (resp., $\bigcup_{U \in \mathcal{U}_0} sclU = 1_X$).

DEFINITION 3.5 [3]. An fts X is said to be fuzzy β-compact space if every fuzzy cover of X by fuzzy β-open sets in X has a finite subcover.

DEFINITION 3.6 [8]. An fts X is said to be fuzzy β-closed if for every fuzzy cover $\mathcal{U}$ of X by fuzzy β-open sets in X, there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $\bigcup_{U\in\mathcal{U}_0} \beta clU = 1X$.

DEFINITION 3.7. An fts X is said to be fuzzy strongly compact [12] (resp., fuzzy P-closed [17]) if every cover of X by fuzzy preopen sets in X has a finite subcover (resp., subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $\bigcup_{U\in\mathcal{U}_0} pclU = 1_X$).

THEOREM 3.8. If X is a fuzzy β-compact space and $f : X \to Y$ is fuzzy β-irresolute surjective mapping, then Y is fuzzy semicompact.

PROOF. Let $\mathcal{V} = \{ V_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of Y by fuzzy semiopen sets of Y. Then as fuzzy semiopen sets are fuzzy β-open, $\mathcal{V}$ is a fuzzy cover of X by fuzzy β-open sets of Y. Now f being fuzzy β-irresolute surjective mapping, $\{f^{-1}(V_\alpha): \alpha \in \Lambda\}$ is a fuzzy cover of X by fuzzy β-open sets of X. As X is fuzzy β-compact, there exists a finite subfamily Λ_0 of Λ such that $\{f^{-1}(V_\alpha): \alpha \in \Lambda_0\}$ also covers X, i.e., $1_X = \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha) \Rightarrow 1_Y = f(1_X) = f\left(\bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha)\right) = ff^{-1}\left(\bigcup_{\alpha\in\Lambda_0} V_\alpha\right) \leq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. Hence Y is fuzzy semicommpact space.

REMARK 3.9. Since fuzzy semicompact space is fuzzy S-closed space, we can state the following theorem.

THEOREM 3.10. If X is fuzzy β-compact space and $f : X \to Y$ is fuzzy β-irresolute surjective mapping, then Y is fuzzy S-closed space.

PROOF. The proof is same as that of Theorem 3.8 and hence omitted.

REMARK 3.11. Since fuzzy preopen set is fuzzy β-open, we can state the following theorem.

THEOREM 3.12. If X is fuzzy β-compact space and $f : X \to Y$ is fuzzy β-irresolute surjective mapping, then Y is fuzzy strongly compact (resp., fuzzy P-closed).

REMARK 3.13. Since for a fuzzy set A in X, $\beta cl\ A\ \leq scl\ A$, $\beta cl\ A\ \leq pcl\ A$, $\beta cl\ A\ \leq cl\ A$, we can easily state the following theorem.

THEOREM 3.15. If X is fuzzy β-closed space and $f: X \rightarrow Y$ is fuzzy β-irresolute surjective mapping, then Y is fuzzy S-closed (resp., fuzzy s-closed, fuzzy P-closed) space.

NOTE 3.16. Instead of space we can state the Theorem 3.8, Theorem 3.10, Theorem 3.12, Theorem 3.14 for a fuzzy set $A\ \in I^X$ also.

References

[1] Abd El-Monsef, M.E., El-Deeb, S.N. and Mahmoud, R.A.; *β-open sets and β-continuous mapping*, Bull. Fac. Sci. Assiut Univ., 12 (1983), 77 – 90.

[2] Azad, K.K.; *On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity*, J. Math. Anal. Appl. 82 (1981), 14 – 32.

[3] Balasubramanian, Ganesan; *On fuzzy β-compact spaces and fuzzy β-extremally disconnected spaces*, Kybernetika 33, No. 3 (1997), 271 – 277.

[4] Chang, C.L.; *Fuzzy topological spaces*, J. Math. Anal. Appl. 24 (1968), 182 – 190.

[5] Fath Alla, M.A.; *On fuzzy topological spaces*, Ph.D. Thesis, Assiut Univ., Sohag, Egypt (1984).

[6] Ganguly, S. and Saha, S.; *A note on compactness in fuzzy setting*, Fuzzy Sets and Systems 34 (1990), 117 – 124.

[7] Ghosg, B.; *Semi-continuous and semi closed mappings and semi-connectedness in fuzzy setting*, Fuzzy Sets and Systems 35 (1990), 345- 355.

[8] Hanafy, I.M.; *Fuzzy β-compactness and fuzzy β-closed spaces*, Turk. J. Math. 28 (2004), 281 – 293.

[9] Mashhour, A.S., Allam, A.A. and Abd El-Hakeim, K.M.; *On fuzzy semicompact spaces*, Bull. Fac. Sci. Assiut Univ., 16(1) (1987), 277 – 285.

[10] Mukherjee, M.N. and Ghosh, B.; *On fuzzy S-closed spaces and FSC-sets*, Bull. Malaysian Math. Soc. (Second Series) 12 (1989), 1- 14.

[11] Mukherjee, M.N. and Sinha, S.P.; *Irresolute and almost open functions between fuzzy topological spac*es, Fuzzy Sets and Systems 29 (1989), 381 – 388.

[12] Nanda, S.; *Strongly compact fuzzy topological spaces*, Fuzzy Sets and Systems 42 (1991), 259 – 262.

[13] Pu, Pao Ming and Liu, Ying Ming; *Fuzzy topology I. Neighbourhood structure of a fuzzy point and Moore-Smith convergence*, Jour. Math. Anal. Appl. 76 (1980), 571 – 599.

[14] Singal, M.K. and Prakash, N.; *Fuzzy preopen sets and fuzzy preseparation axioms*, Fuzzy Sets and Systems 44 (1991), 273 – 281.

[15] Sinha, S.P. and Malakar, S.; *On s-closed fuzzy topological spaces*, J. Fuzzy Math. 2(1) (1994), 95 – 103.

[16] Zadeh, L.A.; *Fuzzy Sets*, Inform. Control 8 (1965), 338 – 353.

[17] Zahran, A.M.; *Strongly compact and P-closed fuzzy topological spaces*, J. Fuzzy Math. 3 (1) (1995), 97 – 102.