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Website: www.aarf.asia Email : editor@aarf.asia , editoraarf@gmail.com

ANALYSIS OF MATCHED ASYMPTOTIC EXPANSIONS METHOD TO PROVIDE APPROXIMATION SOLUTIONS TO DIFFERENTIAL EQUATIONS

Ayele Tulu Yadate[1] , Daba Meshesha Gusu [2]

[1]Lecturer Department of Mathematics, Ambo University, Ambo, Ethiopia

[2] Lecturer Department of Mathematics, Ambo University, Ambo, Ethiopia

ABSTRACT

This study is attempted to introduce some of the basic concepts of asymptotic analysis and provide approximation solutions to differential equations that cannot easily be solved explicitly. The study is divided in to three parts. The first part is the introduction which introduces the back ground of the asymptotic methods of differential equations and some important topics that are essential in the main body. The second part is the main body of the study, which is focusing on perturbation theory and the methods of approximating differential equations. Under perturbation theory, regular perturbation and singular perturbation are discussed using illustrative examples. One of the most common methods in singular perturbation theory; methods of matched asymptotic expansions shall be presented. To demonstrate the theory for this method fully worked introductory examples are considered and the concept of boundary layer, matching, compost expansion, finding further terms, Ven Dyke's matching principle and the location of boundary layer are introduced. The third part is the conclusion of the study which summarizes what we have discussed in the main part of the study. Based on the findings and implication of the study, thus, conclusion and recommendation were forwarded, too.

Key words: *Asymptotic expansions, Boundary layer, Compost expansion, Matched, Perturbation.*

Introduction

When mathematical modeling is used to describe physical, biological or chemical phenomena, one of the most common results is either a differential equation or a system of differential equations, together with appropriate boundary and initial conditions. These differential equations may be ordinary or partial, and finding and interpreting their solution is at the heart of applied mathematics.

The vast majority of differential equations that arise from everyday problems as models for real physical systems cannot easily be solved directly. In these situations we usually have two options. We can use computers to seek complicated numerical solutions or we can look to construct an analytical approximation to the solution using asymptotic expansions. Asymptotic methods have particular importance in many areas of applied mathematics, with the physical problems studied in fluid dynamics providing the main motivation for much of the important development in the subject's history.

One of the oldest and most famous asymptotic results is Stirling's formula, used to approximate *n!* for large values of n,

$$n! \sim (n/e)^n \sqrt{2\pi n}$$

where ~ is used to denote that two functions are asymptotically, or approximately, equivalent. It was Henri Poincare who introduced the term asymptotic expansion during an 1886 paper published in Acta Mathematica, studying irregular integrals of linear equations. He began that paper by analyzing another of Stirling's series: his series for the logarithm of the Euler Gamma Function.

To obtain an approximate solution, it is possible to use an *asymptotic method* if one or smaller dimensionless parameters appear in the differential equation. Moreover, the presence of a small parameter often leads to a *singular perturbation problem*, which can be difficult to solve numerically Thomas [15].

Small, dimensionless parameters usually arise when one physical process occurs much more slowly than another, or when one geometrical length in the problem is much shorter than another. Thus, this paper is focus on the methods applicable to problems presented as differential equations, particularly basics of regular and singular perturbation theory. The method of matched asymptotic expansions for Ordinary differential equations will be considered in detail.

Preliminary Concept

Gauge Functions: If $\lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = k$, for some nonzero constant k, we write $f(\varepsilon) = O(g(\varepsilon))$ for $\varepsilon \ll 1$. We say that f is of order g for small ε. Here g (ε) is a *gauge function*, since it is used to gauge the size of $f(\varepsilon)$. This notation tells us nothing about the constant k. For example, $10^{10} = O(1)$. The order notation only tells us how functions behave as $\varepsilon \to 0$. It is not meant to be used for comparing constants, which are all, by definition, of $O(1)$.

We also have a notation available that displays more information about the behavior of the functions. If $\lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 1$, we write $f(\varepsilon) \sim g(\varepsilon)$, and say that $f(\varepsilon)$ is asymptotic to $g(\varepsilon)$ as $\varepsilon \to 0$.

If the function $\lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0$, we write $f(\varepsilon) = o(g(\varepsilon))$, and say that *f* is much less than *g*.

Asymptotic Sequence: A sequence $\{A_n\}$ for $n = 1, 2, \ . \ . \ .$ is called an asymptotic sequence if for all *n*, $A_{n+1}(z) = o(A_n(z))$ as $z \to z_0$. The order of every term in an asymptotic sequence is less than the order of the previous one.

Asymptotic Expansions: The crucial difference between asymptotic expansions and the power series is that asymptotic expansions need not be convergent in the usual sense Georgescu [5]. Poincare's original definition states that for $A_n(z)$ an asymptotic sequence as $\to z_0$, $\sum_{n=1}^{N} a_n A_n$ is an asymptotic expansion (or asymptotic series) of $f(z)$ if for all *N*, $f(z) = \sum_{n=1}^{N} a_n A_n + o(A_n(z)$ as $z \to z_0$. For all values of *n* the functions $A_n(z)$ are known as the guage functions.

Asymptotic Equivalence: The aim of asymptotic approximation is to find a function that is asymptotically equivalent to the solution of the given problem. Two functions *f* and *g* are asymptotically equivalent, written $f(z) \sim g(z)$ as $z \to z_0$, if $\lim_{z\to z_0} \frac{f(z)}{g(z)} = 1$.

Leading order solutions: leading order solutions of an equation are solutions which are not dependent on small parameter ε where $0 < \varepsilon < 1$.

Boundary layer problems: if ε is a multiplier of the highest derivative or leading term of a polynomial equation (differential equation) ε is known as a boundary layer problem or rarely a matching problem.

Introductory Examples

The Taylor's theorem which is often used within more complicated methods is one of the most useful tools for obtaining asymptotic expansions. Consider the exponential function e^z as an example and the well known Taylor series for $z \to 0$ is given as

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + - - -$$

valid for all z an element of the set of complex numbers. Hence the order of the terms is gradually decreasing for $z \rightarrow 0$. Therefore, from the definition this is clearly a valid asymptotic series and for a 3 term asymptotic expansion we can write

$$e^z = 1 + z + \frac{z^2}{2!}$$

This is known as three term expansion or an expansion to $O(e^3)$ (to order e^3) taken from the highest order of the asymptotic variable the series is expanded to.

Discussions and Results

Perturbation Theory

One of the main uses of asymptotic analysis is to provide approximations to differential equations that cannot easily be solved explicitly. To simplify our discussion we considered only second order ordinary differential equations. However most of the techniques can easily expanded to problems of higher ODEs and PDEs.

Consider the following general second order differential equation for $y(x, \mu)$, which is a function of x,

$$\frac{d^2 y}{dx^2} + p(x, \mu)\frac{dy}{dx} + q(x, \mu)y = r(x, \mu)$$

Where x is an independent variable with respect to which all differentiation and integration is applied. μ and any other variables upon which the solution of y could depend on are known as *physical parameters* and no differentiation or integration is carried out with respect to them.

The variable with respect to which we study the asymptotic behavior is known as the *asymptotic variable*. In classical asymptotic analysis the asymptotic variable is taken as the independent variable of the differential equation. In perturbation theory the asymptotic behavior is studied with respect to a small physical parameter, usually denoted by $\mathcal{E}$. The point in the domain around which the asymptotic behavior is studied is known as the *asymptotic accumulation point*. The most common differential equation problems we focus on in this paper for an approximation are those of perturbation theory, where the accumulation point is $\mathcal{E} = 0$. Perturbation theory deals with problems that contain a small parameter usually denoted by $\mathcal{E}$ and solutions are required as $\mathcal{E}$ approaches 0. Perturbation theory can be divided into *regular* and *singular* forms. The differences between the two will be seen in the coming examples.

Regular Perturbation Problems

The common technique with perturbation problems is to find an expansion with respect to the asymptotic sequence $\{1,\ \mathcal{E},\ \mathcal{E}^2\ .\ .\ .\}$ as $\mathcal{E} \to 0$. The regular (or Poincare) expansion is then

$$u(\mathcal{E}, x) \sim U_0(x) + \mathcal{E}\, U_1(x) + \mathcal{E}_2 U_2(x) + . . . \text{ as } \mathcal{E} \to 0.$$

for gauge functions $U_0, U_1, U_2 \ . . .$ which we will determine.

The solution of ODEs by asymptotic methods often proceeds as follows. We assume that an asymptotic expansion of the solution exists, substitute into the equation and boundary conditions, and equate powers of the small parameter. This determines a sequence of simpler equations and boundary conditions that we can solve.

Example: Consider the initial value problem

$$\frac{d^2y}{dt^2} + \mathcal{E}\,\frac{dy}{dt} + 1 = 0$$

$$y\,(0) = 0; \qquad (1.1)$$

$$\frac{dy}{dt}(0) = 1$$

The above equation represents a projectile motion problem where air friction taken into account. $\mathcal{E} = \frac{kv_0}{mg}$ where k is the coefficient of air friction, g is the gravitational acceleration, m the object's mass and v_0 the initial velocity.

To construct the asymptotic solution when $\mathcal{E} \ll 1$, it can be carried out to find as many terms of the expansion as necessary but in practical situations only a small number of terms are usually needed. Thus, to gives an approximation up to $O\,(\mathcal{E}^2)$, we seek a solution of the form:

$$y(t) = y_0(t) + \mathcal{E}\,y_1(t) + \mathcal{E}^2\,y_2(t) + O\,(\,\mathcal{E}^3) \qquad (1.2)$$

This is now substituted in the differential equation and initial conditions (1.1) to determine functions y_0, y_1 and y_2 and give a 3 term expansion. Of course this Substituting (1.2) in to (1.1) and after rearrange into a hierarchy of powers of $\mathcal{E}$ it gives:

$$\left\{\frac{d^2y_0}{dt^2} + 1\right\} + \mathcal{E}\left\{\frac{d^2y_1}{dt^2} + \frac{dy_0}{dt}\right\} + \mathcal{E}^2\left\{\frac{d^2y_2}{dt^2} + \frac{dy_1}{dt}\right\} + O\,(\mathcal{E}^3) = 0$$

$$y_0(0) + \mathcal{E}\,y_1(0) + \mathcal{E}^2\,y_2(0) + O\,(\,\mathcal{E}^3) = 0$$

$$\frac{dy_0}{dt}(0) - 1 + \mathcal{E}\frac{dy_1}{dt}(0) + \mathcal{E}^2\,\frac{dy_2}{dt}(0) + O\,(\mathcal{E}^3) = 0$$

The next step is then to equate to zero all the terms of each order of $\mathcal{E}$:

$$O(1):\ \frac{d^2y_0}{dt^2} + 1 = 0 \quad \text{where} \quad y_0(0) = 0 \text{ and } \frac{dy_0}{dt}(0) - 1 = 0$$

$$O(\mathcal{E}): \quad \frac{d^2y_1}{dt^2} + \frac{dy_0}{dt} = 0 \quad \text{where} \quad y_1(0) = 0 \text{ and } \frac{dy_1}{dt}(0) = 0$$

$$O(\mathcal{E}^2): \quad \frac{d^2y_2}{dt^2} + \frac{dy_1}{dt} = 0 \quad \text{where} \quad y_2(0) = 0 \text{ and } \frac{dy_2}{dt}(0) = 0$$

Solving these equations gives:

$$y_0(t) = t - \frac{t^2}{2}$$

$$y_1(t) = -\frac{t^2}{2} + \frac{t^3}{6}$$

$$y_2(t) = \frac{t^3}{6} - \frac{t^4}{24}$$

Now putting these into the expansion (1.2) gives an approximation to $O(\mathcal{E}^2)$:

$$\boldsymbol{y(t)} \sim \left(\boldsymbol{t} - \frac{\boldsymbol{t^2}}{\boldsymbol{2!}}\right) + \mathcal{E}\left(-\frac{\boldsymbol{t^2}}{\boldsymbol{2!}} + \frac{\boldsymbol{t^3}}{\boldsymbol{3!}}\right) + \mathcal{E}^2\left(\frac{\boldsymbol{t^3}}{\boldsymbol{3!}} - \frac{\boldsymbol{t^4}}{\boldsymbol{4!}}\right) \qquad (1.3)$$

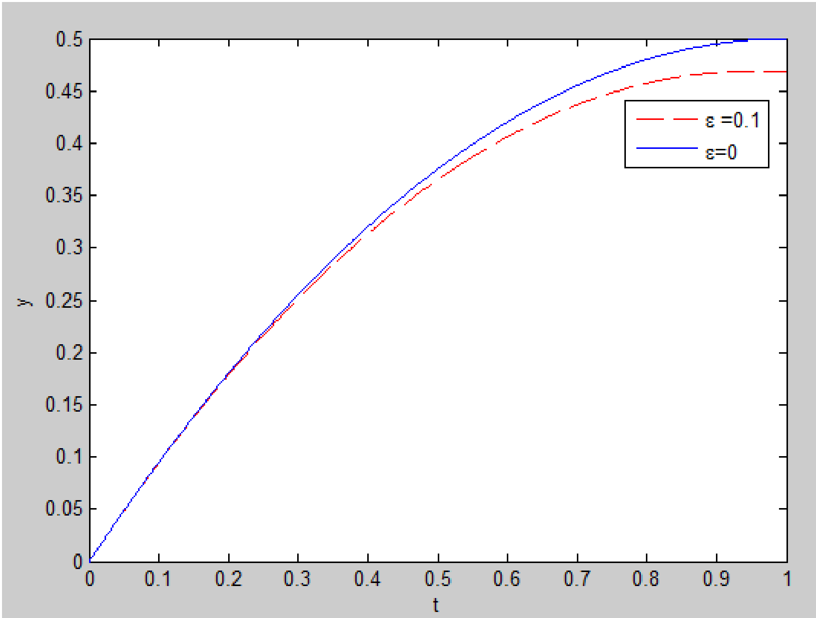

Figure - 1: Asymptotic solutions of (1.1) when $\mathcal{E} = 0$ and $\mathcal{E} = 0.1$

Hence finding an asymptotic expansion is simply a case of assuming a well-known form and substituting it into the problem equation to give an approximate solution. Of course this was an extremely basic example but the fundamental principle is the same wherever regular perturbation techniques can be used. In the next section on singular perturbation theory we will discuss areas where regular perturbation fails.

Singular Perturbation Problem

Singular perturbation theory concerns the study of problems featuring a parameter for which the solutions of the problem at a limiting value of the parameter are different in character from the limit of the solutions of the general problem; namely, the limit is singular. In contrast, for regular perturbation problems, the solutions of the general problem converge to the solutions of the limit-problem as the parameter approaches the limit-value as discussed Cronin [2].

Thus, a perturbation problem is said to be singular when the regular methods produce an expansion that fails, at some point, to be valid over the complete domain. Geometric singular perturbation theory provides a rigorous approach for describing solutions of singularly perturbed dynamical systems, based on Fenichel's analysis of the manifolds underlying the system as discussed elsewhere [2, 7].

There are a number of types of singular perturbation problems that all need different methods to attempt them. We will consider *method of matched asymptotic expansions*. This method is the most common and widely applicable in singular perturbation problem as discussed [10, 14]. When ε is a multiplier of the highest derivative or leading term of a polynomial equation it is known as a boundary layer problem or rarely a matching problem, the reasons for this will become clear in the next section.

Matched Asymptotic Expansions Method

The Method of Matched Asymptotic Expansions has its roots in Ludwig Prandtl's boundary layer theory, developed in 1905 while studying the flow of a viscous incompressible fluid past a wall or body. In physical terms a boundary layer is the layer of fluid at the very border of a flow, against the containing surface for example an aircraft wing or the banks of a river. The thin area here exhibits properties very much different from the rest of the field Bruijn [1].

Introductory Example:

Consider the ordinary differential equation

$$\varepsilon \frac{d^2y}{dx^2} + 2\,\frac{dy}{dx} - y = 0 \qquad (2.1)$$

to be solved for $0 \leq x \leq 1$, subject to the boundary conditions

$$y\,(0) = 0, \quad y(1) = 1.$$

Know we consider an asymptotic solution of the form

$$y(x) = y_0(x) + \varepsilon\, y_1(x) + O(\varepsilon^2). \qquad (2.2)$$

Substituting this into (2.1) gives:

$$2y'_0 - y_0 + \mathcal{E}\,(y''_0 + 2\,y'_1 - y_1) + O\,(\mathcal{E}^{\,2}) = 0 \qquad (2.3)$$

And so the term independent of $\mathcal{E}, O(1)$ is:

$$2y'_0 - y_0 = 0$$

With the general solution

$$y_0(x) = Ae^{x/2} \qquad (2.4)$$

for A an arbitrary constant. This raises the first difficulty: with only one arbitrary constant both boundary condition $y\,(0) = 0$ *and* $y\,(1) = 1$ cannot be satisfied. What this means is that (2.2) and (2.4) cannot provide a valid solution over the whole interval $0 \le x \le 1$ and this is where Prandtl's boundary theory comes into play. It is now assumed that a boundary layer exists at either end of the domain [0, 1] within which the properties of the solutions are very much different, preventing solution (2.4) being valid over the complete interval.

For it assumed that this boundary layer occurs at $x = 0$ and so we will have more than one solution. Inside the boundary layer about $x = 0$ we will have an *inner* or *boundary layer solution* and over the rest of the domain we will have the *outer solution.* Hence (2.4) will be known as first term of the outer solution and it satisfies the second of the two boundary conditions $y\,(1) = 1$.

Using this boundary condition (2.4) which gives as

$$\mathbf{y_0(x) = e^{(x-1)/2}}$$

(2.5)

This is our one term approximation to the solution outside the boundary layer.

At $O(\mathcal{E})$ we have

$$2\,y'_1 - y_1 = -y''_0 = -\frac{1}{4}\,e^{(x-1)/2}$$

to be solved subject to $y\,(0) = 0$ *and* $y\,(1) = 0$. This equation can be solved using an integrating factor, which gives

$$y_1(x) = -\frac{1}{2}\,xe^{(x-1)/2} + \mathrm{B}\,e^{(x-1)/2}$$

for some constant B. Again, we cannot satisfy both boundary conditions, and we just use $y_1(1) = 0$, and evaluate the constant B = ½ to gives

$$\mathbf{y_1(x) = \frac{1}{2}\,(1 - x)\,e^{(x-1)/2}}$$

(2.6)

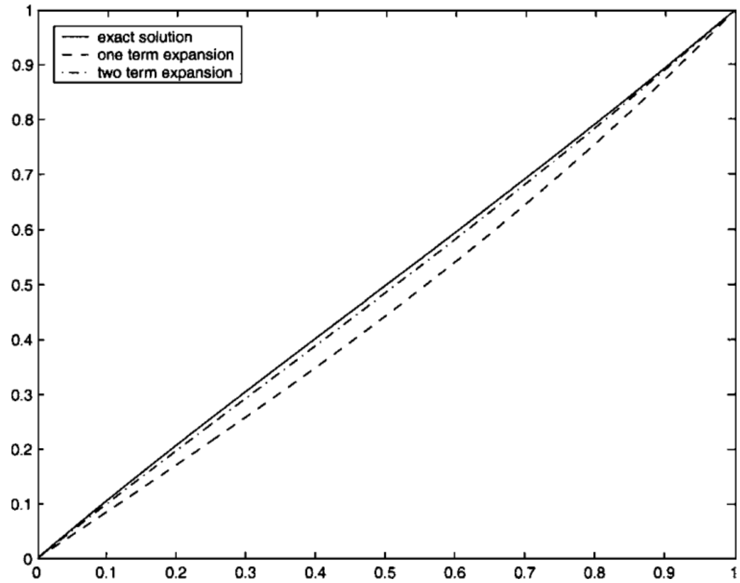

Figure -2: Exact and asymptotic solutions of (2.1) when $\varepsilon = 0.25$. (Source: King, A.C. 2003)

And hence this gives

$$y(x) = e^{(x-1)/2}\left\{1 + \frac{1}{2}\varepsilon\,(1 - x)\right\} + O(\varepsilon^{2})$$

(2.7)

for $\varepsilon \ll 1$.

This shows that $y \to e^{-1/2}\,(1 + \frac{1}{2}\varepsilon)$ as $x \to 0$, which clearly does not satisfy the boundary condition $y(0) = 0$. We must therefore introduce a boundary layer at $x = 0$, across which y adjusts to satisfy the boundary condition.

Boundary Layer

Singularly perturbed differential equations can yield solutions containing regions of rapid variation (rapid compared to the regular length scale for the problem). These regions, which may be apparent in the solution or in its derivatives, are called `*layers*' and often appear at the boundary of the domain (as illustrated in Figure 3). Solutions obtained for the layers (singular distinguished limits) are usually termed *inner solutions* while the slowly varying solutions for the regular distinguished limits are referred to as *outer solutions*.

The uniformly valid solution (composite solution) can be constructed through *asymptotic matching* of the inner and outer solutions, which rely on the fundamental assumption that the different solution forms overlap at on some identifiable region (see Figure 3). Procedures for matching asymptotic expansions have been examined by Kaplun, Van Dyke and others; there are still some fundamental theoretical issues to be resolved as discussed in Eckhaus [4], Kevorkian [8] and Lagerstrom [11].

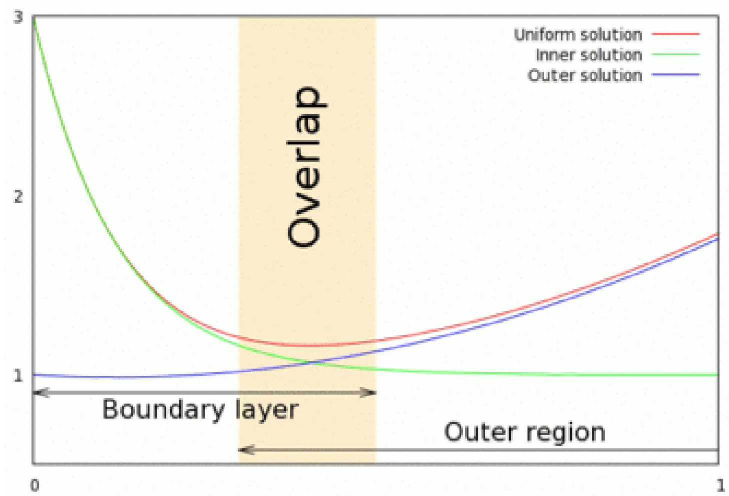

Fig.- 3: Schematic of a boundary layer. (Source Thomas W. & Mark, B. 2009)

The idea is that, in some small neighborhood of $x = 0$, the term $\mathcal{E}y''$, which we neglected at leading order, becomes important because *y* varies rapidly. To deal with the boundary layer at $x = 0$ introduce a ***boundary layer coordinate*** to rescale the problem within the layer;

$$X = \mathcal{E}^{-\alpha} x \tag{2.8}$$

with $\alpha > 0$ (so that $x << 1$) and $X = O(1)$ as $\mathcal{E} \to 0$.

X is also known as the stretching coordinate because under the transformation $x \to X$ with X taken to be fixed, the region becomes much larger as $\mathcal{E} \to 0$. Let $y(x) = Y(X)$ denote the solutions to the problem when using the boundary layer coordinate for $X = O(1)$ and so from (2.8) and the chain rule we get

$$\frac{d}{dx} = \frac{dX}{dx}\frac{d}{dX} = \mathcal{E}^{-\alpha}\frac{d}{dX}$$

and

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left\{\mathcal{E}^{-\alpha}\frac{d}{dX}\right\} = \mathcal{E}^{-2\alpha}\frac{d^2}{dX^2}$$

Using these new coordinates the initial problem (2.1) becomes

$$\mathcal{E}^{1-2\alpha}\frac{d^2Y}{dX^2} + 2\mathcal{E}^{-\alpha}\frac{dY}{dX} - Y = 0; \quad \text{with } Y(0) = 0 \tag{2.9}$$

and the task is to determine the number α.

Since $\alpha > 0$, the second term in this equation is large, and to obtain an asymptotic balance at leading order we must have $\varepsilon^{1-2\alpha} = O(\varepsilon^{-\alpha})$. Which implies that $1 - 2\alpha = -\alpha$, and hence $\alpha = 1$. So (2.8) become $x = \varepsilon X$ and the equation (2.9) is now

$$\frac{d^2Y}{dX^2} + 2\frac{dY}{dX} - \varepsilon Y = 0; \text{ and } Y(0) = 0. \quad (2.10)$$

The region where $\varepsilon << x << 1$ is usually refer to as the *outer region*, with *outer solution* $y(x)$, and the boundary layer region where $x = O(\varepsilon)$ as the *inner region* with *inner solution* $Y(X)$. The other boundary condition is to be applied at $x = 1$. However, $x = 1$ does not lie in the inner region, where $x = O(\varepsilon)$. In order to fix a second boundary condition for (2.10), we will have to make sure that the solution in the inner region is consistent, in a sense that we will make clear below, with the solution in the outer region, which does satisfy $y(1) = 1$.

The appropriate expansion within the boundary layer shall be

$$Y(X) = Y_0(X) + \varepsilon Y_1(X) + O(\varepsilon^2)$$

Substituting this into (2.10) gives

$$\{Y''_0(X) + 2Y'_0(X)\} + \varepsilon\{Y''_1(X) + 2\{Y'_1(X) - Y_0(X)\} + O(\varepsilon^2) = 0 \quad (2.11)$$

At leading order, $Y''_0(X) + 2Y'_0(X) = 0$ to be solved subject to $Y_0(0) = 0$.
The general solution of which is

$$Y_0(X) = A(1 - e^{-2X}) \quad (2.12)$$

for arbitrary constant *A*. At leading order, we now know that

$y \sim e^{(x-1)/2}$ for $\varepsilon << x << 1$ (the outer expansion),

$Y \sim A(1 - e^{-2X})$ for $X = O(1), x = O(\varepsilon)$ (the inner expansion).

Hence *(2.12)* is the solution to the problem within the boundary region at $x = 0$ and we notice that the constant *A* cannot be determined by either of the boundary conditions. *A* must be determined then by the matching process.

Matching of the Inner and Outer Expansions

The important idea is to understand that both the inner and outer expansions are approximations to the same function. Hence where the inner and outer expansions meet up both expansions should provide a valid and equal result. At this stage we already know the outer expansion explicitly. However the inner expansion depends on an unknown constant. The matching process is used to evaluate the constants in the boundary layer expansion and it also plays an important part in forming the composite expansion.

Essentially, as Y_0 leaves the boundary layer ($X \rightarrow \infty$) it must be equal to y_0 as it comes in to the boundary layer, when ($x \rightarrow 0$). From this we obtain that

$$\lim_{X\rightarrow\infty} Y_0 = \lim_{x\rightarrow 0} y_0$$

and hence

$$\lim_{X\rightarrow\infty} A(1 - e^{-2X}) = e^{-1/2}$$

(2.13)

which gives $A = e^{-1/2}$. And so we finally have the first term of our inner expansion

$$Y_0(X) = e^{-1/2}(1 - e^{-2X}) \qquad (2.14)$$

Composite Expansion:

After the first terms of both the inner and the outer expansions obtained; they must be matched together to obtain one composite expansion that approximates the solution over the whole domain [0, 1]. To get the composite expansion the inner and outer expansions are simply added together and the common limit found in (2.13) is subtracted, otherwise it would be included twice in the overlapping region.

So our solution to $O(1)$ is

$$y_c \sim y_0(x) + Y_0(X) - y^{(1,1)} = e^{(x-1)/2} + e^{-1/2}(1 - e^{-2X}) - e^{-1/2}$$

which gives

$$\boldsymbol{y_c} \sim \boldsymbol{e^{(x-1)/2}} - \boldsymbol{e^{-1/2-2x/\varepsilon}} \quad \text{for } 0 \le x \le 1 \text{ } as \text{ } \varepsilon \rightarrow 0$$

(2.15)

A comparison between the one-term inner and outer solutions, composite expansion and the exact solution is given in Figure 4. It should be clear that the inner expansion is a poor approximation in the outer region and vice versa. This composite expansion shows good agreement with the exact solution across the whole domain $0 \le x \le 1$, as expected.

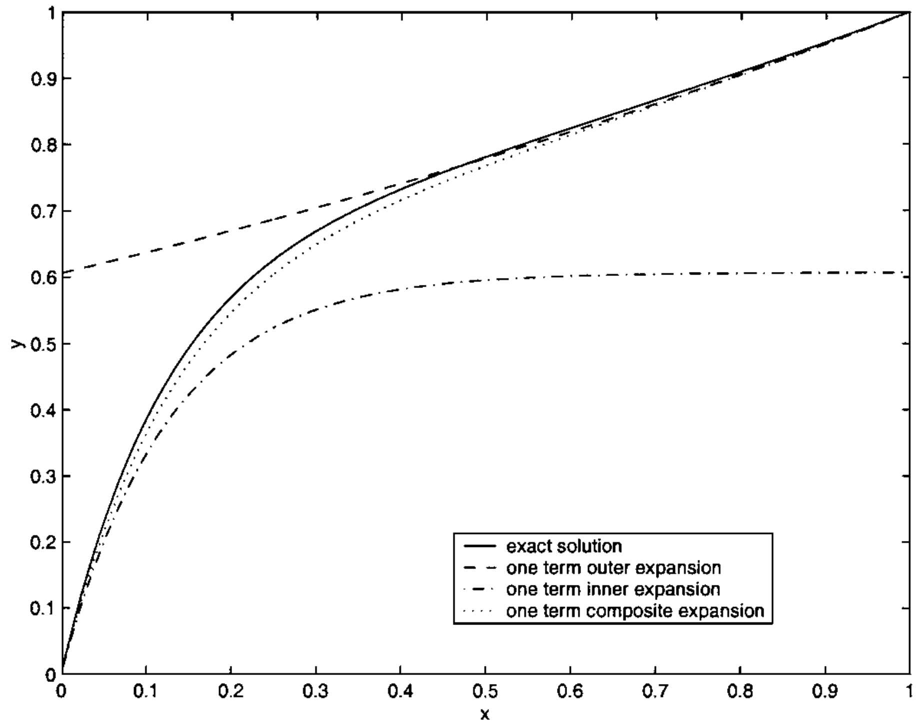

Figure - 4: Exact and asymptotic solutions of (2.1) when $\varepsilon = 0.25$. (Source: King, A.C. 2003)

Further Terms

To calculate a second term for the outer and inner expansions we return to the differential equations obtained using the outer and inner expansions and repeats the same balancing and matching processes. From a second term it can be added into the composite expansion to give a more accurate approximation. Fundamentally this is a simplification of the intermediate matching principle formulated by Saul Kaplan, which introduces the concept of another region in the problem, basically an overlap region where the inner and outer expansions meet and where both will be valid.

We define a new *intermediate variable*, valid over an *overlap domain*, to bridge between the inner and outer regions already defined. In conventional notation this interval is written $[\varepsilon^{\beta_0}, \varepsilon^{\beta_1}]$ and in elementary terms it is the region where both the inner and outer approximations are valid.

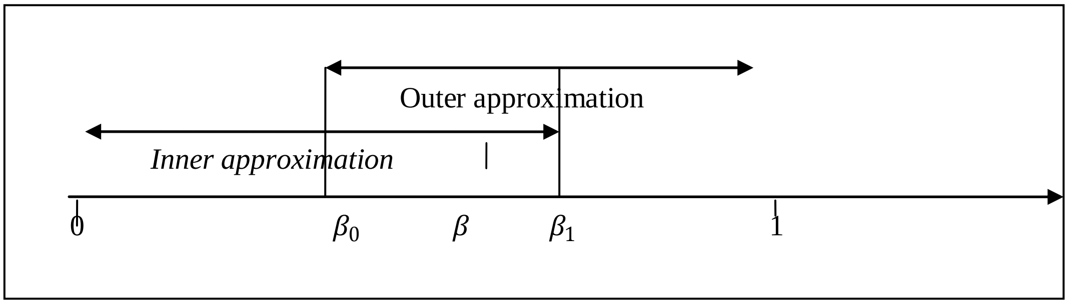

Figure - 5: Overlap regions. Source: (Holmes, M.H. 1995).

Figure 3 shows, a comprehendible illustration similar to Holmes' diagram found in Holmes [6] of how the boundary layer comes into play. In the region around $x = 0$ (at which Ɛ close to **0** valid) and across the newly defined intermediate domain [β_0 , β_1] the inner approximation is valid. The outer approximation is also valid across this interval and over the remaining section of the problem domain, up to $x = 1$.

Figure 3 also gives a good image of how process for obtaining a composite expansion works. The inner and outer expansions are added together and the solution over the intermediate domain is subtracted. Otherwise, if we did not subtract the intermediate solution it would be counted twice as this region is already approximated by both the inner and the outer expansions.

Proving the existence of the intermediate region is not always easy so for our purposes we shall have to bypass the proof, but further study can be found in MacGillivray [12]. With this new overlap domain we now define our intermediate variable as

$$\widetilde{X} = \frac{x}{\varepsilon^{\beta}} \quad \text{with} \quad \varepsilon^{\beta_0} < \varepsilon^{\beta} < \varepsilon^{\beta_1}.$$

Now to use this new matching technique, we should have to find the second term for the outer and inner expansions. For the inner expansion look at (2.12) and on matching the $O(\varepsilon)$ terms we get

$$Y''_1 + 2Y'_1 = Y_0 = A(1 - e^{-2X})$$

Integrating this once gives

$$Y'_1 + 2Y_1 = A\left(X + \frac{1}{2}e^{-2X}\right) + C$$

for some constant C. This can now be solved using an integrating factor, and the general solution to this is

$$Y_1 = \frac{1}{2}A\left(X + \frac{1}{2}e^{-2X}\right) - B(1 - e^{-2X})$$

for some constant A and B, which we need to determine by the matching process. Thus, the two-term asymptotic expansions are:

$$y(x) \sim e^{(x-1)/2} + \left\{\frac{1}{2}\mathcal{E}\,(1 - x)\right\} e^{(x-1)/2} \quad \textit{for } \mathcal{E} \ll x \leq 1$$

$$Y \sim A(1 - e^{-2X}) + \mathcal{E}\left\{\frac{1}{2}A\left(X + \frac{1}{2}e^{-2X}\right) - \mathrm{B}(1 - e^{-2X})\right\} \quad \textit{for } \mathrm{X} = \mathrm{O}(1),\ x = O(\mathcal{E})$$

Now to match the above two expansions together we want to write each in terms of the intermediate variable. That is, we define:

$$x = \mathcal{E}^{\beta}\,\tilde{X} \text{ with } 0 < \beta < 1, \text{ and write } \quad y = \tilde{Y}\,(\tilde{X}).$$

(2.16)

Hence in an intermediate region or overlap region, they should be equal where $\mathcal{E} \ll x \ll 1$. And we expect in such a region *both* expansions to be valid.

In terms of the intermediate variable, $\tilde{X}$, the outer expansion becomes

$$y(x) \sim e^{-\frac{1}{2}}\exp\left(\frac{1}{2}\mathcal{E}^{\beta}\,\tilde{X}\right) + \frac{1}{2}\mathcal{E}\left(1 - \mathcal{E}^{\beta}\tilde{X}\right) e^{-\frac{1}{2}}\exp\left(\frac{1}{2}\mathcal{E}^{\beta}\,\tilde{X}\right) \text{ for } \mathcal{E} \ll x \leq 1$$

when $\tilde{X} = \mathrm{O}(1)$, we can expand the exponentials as Taylor series, and gives

$$\tilde{Y} \sim e^{-\frac{1}{2}}\left(1 + \frac{1}{2}\mathcal{E}^{\beta}\,\tilde{X}\right) + \frac{1}{2}\mathcal{E}e^{-\frac{1}{2}} + \mathrm{O}(\mathcal{E}) \qquad (2.17)$$

Since $\mathrm{x} = \mathcal{E}X = \mathcal{E}^{\boldsymbol{\beta}}\,\tilde{\mathrm{X}}$ we get $\mathrm{X} = \mathcal{E}^{\boldsymbol{\beta}-\mathbf{1}}\,\tilde{\mathrm{X}}$, the inner expansion is

$$\tilde{Y} \sim A\left\{1 - \exp\left(-2\mathcal{E}^{\beta-1}\tilde{X}\right)\right\}$$

$$+\,\mathcal{E}\left[\frac{1}{2}A\mathcal{E}^{\beta-1}\tilde{X}\left\{1 + \exp\left(-2\mathcal{E}^{\beta-1}\tilde{X}\right)\right\} - \mathrm{B}\left\{1 - \exp\left(-2\mathcal{E}^{\beta-1}\tilde{X}\right)\right\}\right]$$

Since for all $n > 0$, $\exp\left(-2\mathcal{E}^{\beta-1}\tilde{X}\right) = \mathrm{O}(\mathcal{E}^{n})$ (It is exponentially small for $\beta < 1$), we have

$$\tilde{Y} \sim A + \frac{1}{2}A\mathcal{E}^{\beta}\,\tilde{X} - \mathrm{B}\mathcal{E} + O(\mathcal{E})$$

(2.18)

Since by the process of asymptotic matching (4.17) and (4.18) must be identical, we need $A = e^{-1/2}$, and we also get $\mathrm{B} = -\frac{1}{2}e^{-1/2}$. The inner and outer expansions are known as *matched asymptotic expansions*.

Van Dyke's Matching Principle

The use of an intermediate variable in an overlap region can get *very* tedious in more complicated problems. *Van Dyke's matching principle* is a method that works most, but not all, of the time, and it is much easier to use. *To* use this method let's start with the explanation. Let

$$y(x) = \sum_{n=0}^{N} \varphi_n(\mathcal{E}) y_n(x)$$

be the outer expansion and

$$Y(\mathrm{X}) = \sum_{n=0}^{N} \psi_n(\mathcal{E}) Y_n(\mathrm{X})$$

be the inner expansion with respect to the asymptotic sequences $\varphi_n(\mathcal{E})$ and $\psi_n(\mathcal{E})$, with $x = f(\mathcal{E})X$.

In order to examine how the inner expansion behaves in the outer region, we can write $Y(X)$ in terms of *x* and retain *M* terms in the resulting asymptotic expansion. We denote this by $y^{(N,M)}$, the M^{th} order outer approximation of the inner expansion. Similarly, to examine how the outer expansion behaves in the inner region we can write $y(x)$ in terms of $X = {}^{x}/_{f(\mathcal{E})}$, and retain *M* terms in the resulting expansion. We denote this by $Y^{(N,M)}$, the M^{th} order inner approximation of the outer expansion. Van Dyke's matching principle states that

$$y^{(N,M)} = Y^{(N,M)}.$$

Let's see how this is valid for our example:-

In terms of the outer variable, the inner expansion is:

$$Y \sim A\{1 - \mathrm{e}^{-2x/\mathcal{E}}\} + \mathcal{E}\left[\frac{1}{2} A \frac{x}{\mathcal{E}}\{1 + \mathrm{e}^{-2x/\mathcal{E}}\} - \mathrm{B}\{1 - \mathrm{e}^{-2x/\mathcal{E}}\}\right]$$

$$\sim Y^{(2,2)} = A + \frac{1}{2}Ax - B\mathcal{E} \text{ for } x = O(1)$$

In terms of the inner variable, the outer expansion is:

$$y(x) \sim \exp\left(-\frac{1}{2} + \frac{1}{2}\mathcal{E}X\right) + \frac{1}{2}\mathcal{E}(1 - \mathcal{E}X)\exp\left(-\frac{1}{2} + \frac{1}{2}\mathcal{E}X\right)$$

$$\sim y^{(2,2)} = \mathrm{e}^{-\frac{1}{2}}\left(1 + \frac{1}{2}\mathcal{E}X + \frac{1}{2}\mathcal{E}\right) \text{ for } X = O(1)$$

In terms of the outer variable: $y^{(2,2)} = \mathrm{e}^{-\frac{1}{2}}\left(1 + \frac{1}{2}x + \frac{1}{2}\mathcal{E}\right)$

Since Van Dyke's matching principle states that $Y^{(2,2)} = y^{(2,2)},$ this gives $A = e^{-1/2}$ and $\mathrm{B} = -\frac{1}{2}e^{-1/2}$ rather more painlessly than before. $x = x_0$ *for* $0 < x_0 < 1$.

Note that, in terms of Van Dyke's matching principle, we can write the composite solution of any order as

$$y \sim y_c(N,M) = \sum_{n=0}^{M} y_n(x) + \sum_{n=0}^{N} Y_n(X) - \ y(N,M)$$

In general the method of matched asymptotic expansions can be useful for differential equations with an Ɛ coefficient multiplying the highest order derivative. Frequently these hold a boundary layer preventing the whole set of boundary conditions being satisfied by a regular perturbation solution. Where the regular solution fails we introduce new coordinates to describe the solution inside the boundary layer and produce two separate approximations valid over different sections of the domain. These solutions must be matched together and combined to form a single expansion applicable generally found in King [9].

The Location of the Boundary Layer

In our example, when we constructed the outer solution, it was identified that a boundary layer exists at either $x = 0$ or $x = 1$ and we assumed it was located at $x = 0$. There is no clear and enough information on identifying the location of a boundary layer. According to Murray [13] writes identifying the layer location comes from experience' while Van Dyke [16] method makes use of the exact solutions to the problems to determine the layer location, but it is not much work for a problem that cannot easily be solved exactly.

In our pervious problems, if we assume that there is a boundary layer at $x = x_0$. That is, if $x_0 \neq 0$ and $x_0 \neq 1$ this is an *interior layer*. And we would need to use a more general boundary layer coordinate in (2.16). In general the transformation to use would be

$$X = \frac{x - x_0}{\varepsilon^{\alpha}} \ \textit{and} \ y(x) = Y(X)$$

where x_0 represents the location of the layer with $\alpha > 0$ 1 and $Y(X) = O(1)$ for $\varepsilon \ll 1$.

As before, we find that we can only obtain an asymptotic balance at leading order by taking $\alpha = 1$, so that $x = x_0 + \varepsilon X$ and

$$\frac{d^2Y}{dX^2} + 2\frac{dY}{dX} - \varepsilon = 0$$

At leading order, as before, $Y_0 = A_0 + B_0 e^{-2X}$. As $X \to -\infty$, Y_0 becomes exponentially large, and cannot be matched to the outer solution. This forces us to take $x_0 = 0$, since then this solution is only needed for $X \gg 0$, and, as we have seen, we can construct an asymptotic solution.

In our example we had simple layer at one of the boundaries of the interval, but much more complex layer dependence is possible. Layers can randomly locate in the middle of the domain which makes problems immediately more difficult as the location is not always easy to determine

White [17]. It is even possible to have nested layers, where we discover that there are two possibilities in choosing ε^{α} in the layer coordinate (2.16).

Boundary layer problems are one of the most common of all asymptotic problems. The Navier-Stokes equations governing fluid flow at high Reynolds number are the typical example dealt with widely by Van Dyke [16] and many other applied mathematicians. More recent and very much still developing adaptations of singular perturbation theory have appeared in financial mathematics with some examples concerning small limits of volatility in dealing with the Black-Scholes option pricing model Duck [3].

Conclusion

To conclude our study it is important to emphasize how small a portion of asymptotic analysis we have covered here. We have introduced the concepts from the original definitions of asymptotic analysis and gone forward to use them in examples from perturbation theory.

Perturbation theory is one of the most important methods of approximation due to its strong historical connection with physics, particularly the area of fluid mechanics. The previously mentioned examples in financial engineering confirm that today perturbation theory is still important as it was during the development stages and in fact new areas where perturbation techniques are useful are still to be found.

The method we met here, matched asymptotic expansions, is the most widely useful and fundamentally important to solve differential problems, this method require the introduction of new coordinates and matching of asymptotic expansions and we can apply for both ODEs and PDEs. For this reason the matched asymptotic expansions could be considered the more basic and more reliable method.

In this paper we simplified our examples by considering only ODEs up to second order. The same methods used can be easily adapted to solve differential equations of much higher orders. With slightly trickier adaptation they can also be extended into PDEs, and this is vastly important in fluid dynamics.

To extend our study in the areas we have crossed we could further investigate the method of matched asymptotic expansions. What we have seen was a simple case where layers existed at one of the boundaries and the layer was easy to locate. In reality there may be further complications where the location of a layer can be difficult to define or we can have several layers all present to complicate a single problem.

Finally, based on the aforementioned, discussions and conclusions, It recommended that the study can be repeated in this area and further investigate when several layers all present in a single problem and also the study can be extended into PDEs, as this is very much important in fluid dynamics.

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