ON CERTAIN GENERALIZED VOLTERRA –TYPE INTEGRAL EQUATIONS WITH QUASI-POLYNOMIAL FREE-TERM

B.B. Jaimini,
Department of Mathematics, Govt. P.G. College, Kota-324001, (Rajasthan), India.

Hemlata Saxena,
Department of Mathematics, Career Point University, Kota (Rajasthan), India.

ABSTRACT

In this paper we investigate the solution of certain generalized Volterra-type integral equation with Quasi polynomial free term involving the generalized G-function of several variables. The equation is solved by the method of Laplace Transform. The results obtained are of general character and provide generalizations of the results given by Srivastava and Saxena (2005) and Saxena and Kalla (2005).

Keywords : Volterra integral equation , Generalized G and R-Functions, Laplace Transform, Riemann-Liouville Fractional Integral Operator, Mittag-L effler function.

MSC (2010) : 33 C 70 ,44 A 10, 45 D 05, 26 A 33

1. Introduction and Preliminaries

We propose the following generalizations of the G and R -functions studied by [Hartely and Lorenzo (1999), (2000)].

$$G_{(q),\delta,(\gamma)}\left[a_1,z;...;a_n,z\right]=z^{\sum_{i=1}^{n}q_i\gamma_i-\delta-1}$$

$$\sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{i=1}^{n} \left[\frac{(\gamma_i)_{k_i} \left(a_i z^{q_i}\right)^{k_i}}{\Gamma(1+k_i)} \right] \frac{1}{\Gamma\left(\sum_{i=1}^{n} q_i \gamma_i - \delta + \sum_{i=1}^{n} k_i q_i \right)} \quad .....(1)$$

at $\gamma_i = 1 \forall (i = 1, 2, ..., n)$ and $z \to z - c$, it reduces into generalized R-function defined as follows:

$$R_{(q),\delta}[a_1, ..., a_n; c, z] = (z-c)^{\sum_{i=1}^{n} q_i - \delta - 1}$$

$$\sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{i=1}^{n} \left[\frac{\left[a_i (z-c)^{q_i} \right]^{k_i}}{\Gamma\left(\sum_{i=1}^{n} q_i - \delta + \sum_{i=1}^{n} k_i q_i \right)} \right] \quad .....(2)$$

In the definitions (1) and (2), the symbol $(q) \equiv q_1, ..., q_n$ and $(\gamma) \equiv \gamma_1, ..., \gamma_n$ will be employed

Integral Transforms

The Laplace transform of $f(t)$ is defined by

$$L\{f(t) : p\} = \int_0^{\infty} f(t) e^{-pt} dt = F(p), \ \text{Re}(p) > 0 \quad .....(3)$$

The Laplace transform of the operator ${}_0D_t^{-\nu}$ in view of above definition is given by [Erdelyi et al. (1954)] :

$$L\left[{}_0D_t^{-\nu} f(t); p \right] = p^{-\nu} F(p) \quad .....(4)$$

where ${}_0D_t^{-\nu} f(t)$ is the Riemann- Liouville fractional integral operator defined as follows [Miller and Ross (1993), Oldhem and Spanier (1974), Samko et al.(1993)].

$$ {}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u)\,du \,,\ \operatorname{Re}(\nu) > 0. \qquad .....(5)$$

The Laplace transform of generalized *G*- function is obtained in the form

$$L\left\{G_{(q),\delta,(\gamma)}\left[a_1^{q_1}, z;\ldots;a_n^{q_n}, z\right]; s\right\} = s^{\delta-\sum_{i=1}^{n}\gamma_i q_i} \prod_{i=1}^{n}\left[1-\left(\frac{s}{a_i}\right)^{-q_i}\right]^{-\gamma_i} \qquad .....(6)$$

by making use of the binomial series , defined by

$$(1-x)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_r x^r}{r!} \qquad .....(7)$$

and applying the result

$$L^{-1}\left\{p^{-\rho}; t\right\} = \frac{t^{\rho-1}}{\Gamma(\rho)} \,,\ Re(p) > 0 \,,\ Re(\rho) > 0 \,. \qquad .....(8)$$

The object of this paper is to investigate the solution of a generalized Volterra-type integral equation with quasi polynomial free term . The method followed is of Laplace transform.

2. Main Theorem

Theorem-1

For the Quasi-polynomial [Kilbas et al. (2006)] in free term the following integral equation with generalized G-function of several variables defined in (1),

$$D_{\tau}^{-\nu}h(\tau)=\lambda\int_{0}^{\tau}G_{(q),\delta,(\gamma)}\left[\omega_1^{q_1},(\tau-\xi);\ldots;\omega_n^{q_n},(\tau-\xi)\right]h(\xi)\,d\xi+\mu\sum_{r=1}^{l}f_r(\tau)^{\rho_r}$$

.....(9)

has its solution in the following form

$$h(\tau)=-\mu\sum_{k=0}^{\infty}\lambda^{-k-1}\int_{0}^{\tau}\left[\frac{d^m}{d\xi^m}\left(\sum_{r=1}^{l}f_r(\xi)^{\rho_r}\right)\right].$$

$$G_{(q),\delta(1+k)+\nu k+m,(\gamma)(1+k)}\left[\omega_1^{q_1},(\tau-\xi);\ldots;\omega_n^{q_n},(\tau-\xi)\right]d\xi \qquad .....(10)$$

where for $i=1,\ldots,n;\nu,q_i,\delta,\gamma_i,\omega,\lambda\in C\ \left(\operatorname{Re}(\nu),\operatorname{Re}q_i,\operatorname{Re}(\delta)>0\right)$ and $f\in L(a,b)$.

Proof of (10)

To obtain the solution of the integral equation (9), we first apply the Laplace transform on both sides of (9), then in view of the results (4) and (6) and using the convolution theorem we get

$$s^{-\nu}H(s)=\lambda(s)^{\delta-\sum_{i=1}^{n}\gamma_i q_i}\prod_{i=1}^{n}\left\{\left[1-\left(\frac{s}{\omega_i}\right)^{-q_i}\right]^{-\gamma_i}\right\}H(s)+\mu\sum_{r=1}^{l}F_r(s)^{\rho_r}$$

where $H(s)$ and $F_r(s)^{\rho_r}$ are the Laplace transforms of $h(\tau)$ and $f_r(\tau)^{\rho_r}\ (r=1,\ldots,l)$.

Solving for $H(s)$,we see that

$$H(s)=\mu\sum_{r=1}^{l}F_r(s)^{\rho_r}\left[s^{-\nu}-\lambda(s)^{\delta-\sum_{i=1}^{n}\gamma_i q_i}\prod_{i=1}^{n}\left\{\left[1-\left(\frac{s}{\omega_i}\right)^{-q_i}\right]^{-\gamma_i}\right\}\right]^{-1}$$

which on simplifying gives

$$H(s) = -\mu\lambda^{-1}(s)^{\sum_{i=1}^{n}\gamma_i q_i - \delta} \prod_{i=1}^{n}\left\{\left[1-\left(\frac{s}{\omega_i}\right)^{-q_i}\right]^{\gamma_i}\right\}\left\{\sum_{r=1}^{l}F_r(s)^{\rho_r}\right\}$$

$$\left[1-\lambda^{-1}(s)^{\sum_{i=1}^{n}\gamma_i q_i - \delta - \nu}\prod_{i=1}^{n}\left\{1-\left(\frac{s}{\omega_i}\right)^{-q_i}\right\}^{\gamma_i}\right]^{-1},$$

$$\left|\lambda^{-1}(s)^{\sum_{i=1}^{n}\gamma_i q_i - \delta - \nu}\prod_{i=1}^{n}\left\{1-\left(\frac{s}{\omega_i}\right)^{-q_i}\right\}^{\gamma_i}\right| < 1$$

in view of binomial expansion we obtain the following form now

$$H(s) = -\mu\sum_{k=0}^{\infty}\lambda^{-k-1}(s)^{\sum_{i=1}^{n}\gamma_i q_i(1+k)-\delta(1+k)-\nu k - m}$$

$$\prod_{i=1}^{n}\left\{\left[1-\left(\frac{s}{\omega_i}\right)^{-q_i}\right]^{\gamma_i(1+k)}\right\}\left[\sum_{r=1}^{l}s^m F_r(s)^{\rho_r}\right], \quad \left|\left(\frac{s}{\omega_i}\right)^{-q_i}\right| < 1$$

for all (i = 1, …, n).

Now on taking inverse Laplace transform of both sides we atonce arrive at the desired result (10).

3. Special Cases

(i) If we set $\gamma_i = 1(i = 1,...,n)$, it reduces to the results involving R-function of several variables, defined in (2), as in the form of following Corollary-1

Corollary-1

The integral equation

$$D_t^{-\nu} h(\tau) = \lambda \int_0^\tau R_{(q),\delta}\left[\omega_1^{q_1}, 0, (\tau-\xi); \ldots; \omega_n^{q_n}, 0, (\tau-\xi)\right] h(\xi)\, d\xi + \mu \sum_{r=1}^{l} f_r(\tau)^{\rho_r}$$

.....(11)

has its solution given by

$$h(\tau) = -\mu \sum_{k=0}^{\infty} \lambda^{-k-1} \int_0^\tau \left[\frac{d^m}{d\xi^m} \left(\sum_{r=1}^{l} f_r(\xi)^{\rho_r} \right) \right.$$

$$\left. G_{(q),\delta(1+k)+\nu k+m,(1+k)}\left[\omega_1^{q_1}, (\tau-\xi); \ldots; \omega_n^{q_n}, (\tau-\xi)\right] d\xi \right] \quad .....(12)$$

where for $i = 1, \ldots, n$ $\nu, q_i, \delta, \omega, \lambda \in C$ $\left(\mathrm{Re}(\nu), \mathrm{Re}\, q_i, \mathrm{Re}(\delta) > 0\right)$ and $f \in L(a,b)$.

5

(ii) If we take $l = 1, \rho_1 = 1$ and $\sum_{i=1}^{n} \gamma_i q_i - \delta = \sigma; \omega_i^{q_i} \to w_i$ then (9), (10) s

reduce to the following results given by Saxena and Kalla (2005):

Corrollary-2

The Volterra type integral equation

$$D_\tau^{-\upsilon} h(\tau) = \lambda \int_0^\tau (\tau-\xi)^{\sigma-1} h(\xi)$$

$$F^{0:1;\ldots;1}_{1:0;\ldots;0}\left[\begin{matrix} - & :(\gamma_1,1),\ldots,(\gamma_n,1); \\ (\sigma:q_1,\ldots,q_n): & -\ ,\ldots,\ - \ ; \end{matrix}\ \omega_1(\tau-\xi)^{q_1}, \ldots, \omega_n(\tau-\xi)^{q_n}\right] d\xi + \mu f(\tau)$$

….(13)

has its solution given explicitly by

$$h(\tau) = -\mu \sum_{r=0}^{\infty} \lambda^{-r-1} \int_0^{\tau} (\tau-\xi)^{m+\upsilon r-\sigma r-\sigma-1} \left(\frac{d^m}{d\xi^m} f(\xi) \right)$$

$$F^{0:1;\dots;1}_{1:0;\dots;0} \left[{}^{-}_{(m+\upsilon r-\sigma r-\sigma-1:q_1,\dots,q_n):} {}^{:(-\gamma_1(r+1),1),\dots,(-\gamma_n(r+1),1);}_{-\quad ,\dots, \quad -\quad ;} \omega_1(\tau-\xi)^{q_1},\dots,\omega_n(\tau-\xi)^{q_n} \right] d\xi$$

….(14)

$0 \le \mathrm{Re}(\sigma) < \min\{m, \upsilon\}; \lambda, \mu \in C, \upsilon \ge 0, f \in C^m[0,\infty);$
$f^j(0) = 0, \mathrm{Re}(q_j) > 0 (j = 0,1\dots, m-1), m \in N$

(iii) If we take $l_1 = 1, \rho_1 = 1$, $\sum_{i=1}^{n} \gamma_i q_i - \delta = \sigma; \omega_i^{q_i} \to w_i$ and $q_1 = \dots = q_n = 1$,

then in view of the known result given by Saxena and Kalla (2005) the result (9), (10) reduce to the following results given by Srivastava and Saxena (2005) :

Corrollary-3

The Volterra type integral equation

$$D_\tau^{-\upsilon} h(\tau) = \frac{\lambda}{\Gamma(\sigma)} \int_0^{\tau} (\tau-\xi)^{\sigma-1} h(\xi) \ \phi_2^{(n)} \left[\gamma_1,\dots,\gamma_n; \sigma; \omega_1(\tau-\xi) ,\dots, \omega_n(\tau-\xi) \right] d\xi$$

$$+ \ \mu f(\tau) \qquad \text{….(15)}$$

has its solution given explicitly by

$$h(\tau) = -\mu \sum_{r=0}^{\infty} \lambda^{-r-1} \int_0^{\tau} \frac{(\tau-\xi)^{m+\upsilon r-\sigma r-\sigma-1}}{\Gamma(m+\upsilon r-\sigma r-\sigma)} \left(\frac{d^m}{d\xi^m} f(\xi) \right)$$

$$\phi_2^{(n)} \left[-\gamma_1(r+1),\dots,-\gamma_n(r+1); m+\upsilon r-\sigma r-\sigma; \omega_1(\tau-\xi) ,\dots, \omega_n(\tau-\xi) \right] d\xi$$

….(16)

$0 \le \mathrm{Re}(\sigma) < \min\{m, \upsilon\}; \lambda, \mu \in C, \upsilon \ge 0, f \in C^m[0,\infty); f^j(0) = 0 (j = 0,1\dots, m-1), m \in N$

4. Conclusion:

We conclude the present paper by passing the remark that the integral equations studied in the paper involve the generalized G- and R-function of several variables in the kernel, with a quasi polynomial free term containing a continuous function $f(\tau)$. These functions are very general in nature and may reduced into many special functions of Mittag-Lefller type of one and several variables. The Mittag-Leffler function is very important function as it arises in the solution of fractional order differential and integral equations which appears in various problems of applied physics, engineering and mathematical sciences. Another aspect of the present study lies in the fact that, if we consider the case when $0 < \upsilon < 1$, all the results studied in the paper may reduced for differential equations and the corresponding solutions , we skip the details here.

***Acknowledgement*:**

The authors are grateful to Prof. R. K. Saxena, Professor Emeritus, Department of Mathematics, JNV University , Jodhpur (India) for valuable suggestions to improve this paper which lead the paper in the present form.

REFERENCES

1. Erdelyi , A., Magnus, W. , Oberhettinger, F. and Tricomi, F. G. (1954). *Tables of Integral Transforms*, Vol. – II, McGraw –Hill, New York Toronto &London.
2. Kilbas, A.A., Srivastava, H. M. , Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential equations*, North – Holland Mathematics studies 204, Elsevier, London.
3. Lorenzo, C. F. and Hartely, T. T. (1999). Generalized functions for the fractional calculus, NASA Tech. Pub. – 209424 , pp.1-17.
4. Lorenzo, C. F. and Hartely, T. T. (2000). R-function relationships for application in the fractional calculus, NASA Tech. Mem. 210361, pp.1-22.
5. Miller, K. S. and Ross, B. (1993) : *An Introduction to the Fractional Calculus and Fractional Differential Equation,* John Wiley & Sons, New York .
6. Oldhem, K. B. and Spanier, J. (1974). The fractional calculus : *Theory and application of differential and integration to arbitrary order*. Academic Press, New York .

7. Samko, S.G., Kilbas, A. A. and Marichev, O. I. (1993). *Fractional integrals and derivatives : Theory and applications*, Gordon and Breach Science Publisher, New York .
8. Saxena, R. K. and Kalla, S.L. (2005). Solution of Volterra-type integro differential equations with a generalized Lauricella confluent
9. hypergeometric function in the kernels, International Journal of Mathematics and Mathematical Sciences :Vol.8, pp. 1155-1170.
10. Srivastava, H. M. and Saxena, R. K. (2005). Some Volterra-type fractional integro- differential equations with a multivariate confluent hypergeometric function as their kernels , J. Integral Equations with Applications , Vol.17, No. 2 , pp.199-217.