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PARAMETER-DEPENDENT FUNCTIONAL FORMS OF DETERMINING MATRICES FOR A CLASS OF TRIPLE-DELAY CONTROL SYSTEMS

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ABSTRACT

This paperobtained functional forms of determining matrices for certain pertinent parameter ranges, thus bridging the knowledge gap in this area of acute research need. The proofs were achieved by the exploitation of key facts about permutations, combinations of summation notations, change of variables techniques and the compositions of sigma and max functions.

KEYWORDS: Delay, Determining,Functional, Permutation, System, Triple

1. INTRODUCTION

The theory of dynamical systems is motivated by a desire to exceed the simple stage of computing particular solutions of models to establishing various structural relations among certain parameters and their influence on the solutions. The importance of the structural exploration derives from the fact that it serves as a clue into the system behaviour. This enables one to roughly outline the solution of a complex system, which is a spring board for creatively refining the original model.

Secondly, one can circumvent the arduous task of explicitly examining numerous particular solutions by leveraging on structural exploration. For example, the stability of complex economic processes of fuel price adjustments can often be inferred from their structural forms. Controllability is a very important concept that is applied in aerospace engineering, optimal control theory, systems theory, quantum systems, power systems, industrial and chemical process controls etc. This concept was first introduced by Kalman in 1963 (Zmood, 1971) and a survey of controllability of dynamical systems was done by Klamka (2013). In addition, there have been intense research activities on qualitative approach to controllability of linear and nonlinear systems. Xianlong (2013) researched on approximate controllability of semi linear neutral retarded systems and Jackreece (2014) worked on the controllability of neutral integro-differential equations.

There has been a flurry of research activities by Control Theorist and Applied Mathematicians on the subject of controllability of functional differential control systems in recent years (Xue & Yong, 2016). However, one is not aware of any other results that comprehensively interrogated the controllability of linear autonomous control systems of single-delay neutral and double-delay types via the structures of the determining matrices except (Ukwu, 2014b; 2016).

Furthermore, in the study of Euclidean controllability of linear autonomous control systems, determining matrices are preferred veritable tools as they are the least computationally intensive when compared to indices of control systems matrices or controllability Grammians. The determining matrices have computational advantage over indices of control systems or controllability Grammians due to the fact that they offer considerable savings in computational time when deployed in the investigating of the Euclidean controllability of systems.

Unfortunately there is no known published work that has attempted the extension of the great feats of (Ukwu, 2014b; 2016) to delay control systems with triple time-delays in the state variables until (Ukwu & Temuru, 2018). This could be attributed to the severe difficulty in identifying recognizable mathematical patterns needed for any conjecture on functional forms of determining matrices and subsequent inductive proof. It is against this backdrop that this article makes a positive contribution to knowledge by correctly establishing relevant results on functional forms of determining matrices with respect to the afore-mentioned triple-delay systems for certain pertinent parameters.

2. THEORETICAL UNDERPINNING

2.1 Identification of Work-Based Triple-delay Linear Autonomous Control Systems

Consider the triple-delay linear autonomous control system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h) + A_3 x(t-3h) + Bu(t); t \geq 0 \qquad (1)$$

$$x(t) = \phi(t), t \in [-3h, 0], h > 0 \qquad (2)$$

where A_0, A_1, A_2 and A_3 are $n \times n$ constant matrices with real entries, B is an $n \times m$ constants matrix with real entries. The initial function ϕ is in $C([-3h,0],\mathbf{R}^n)$, the space of continuous functions from $[-3h,0]$ into the real n-dimensional Euclidean space, $\mathbf{R}^n$ with norm defined by $\|\phi\| = \sup_{t\in[-3h,0]}\{|\phi(t)|\}$, (the sup norm). The control u is in the space $L_\infty([0,t_1],\mathbf{R}^n)$, the space of essentially bounded measureable functions taking $[0,t_1]$ into $\mathbf{R}^n$ with norm $\|\phi\| = ess\sup_{t\in[0,t_1]}\|u(t)\|$.

Any control $u \in L_\infty([0,t_1],\mathbf{R}^n)$, will be referred to as an admissible control.

See Chidume (2007) for further discussion on $L_p(\text{or } L^p), p \in \{1,2,\cdots,\infty\}$.

(Ukwu &Temuru, 2018)obtained the following preliminary and major results on the functional form of the determining matrices of the system (1) for some parameters, as well as on the j- interval $[3k-3,\infty)$.

Their results are as follows:

Let r_a, r_b, r_c be nonnegative integers and let $P_{a(r_a),b(r_b),c(r_c)}$ denote the set of all permutations of

$\underbrace{a,a,\ldots,a}_{r_a \text{ times}}\ \underbrace{b,b,\ldots,b}_{r_b \text{ times}}\ \underbrace{c,c,\ldots,c}_{r_c \text{ times}}$: the permutations of the objects $a,b,c,$ in which i appears r_i times, $i \in \{a,b,c\}$.

2.1.1 **Determining Equations:** Uniqueness and Existence

Let $Q_k(s)$ be an $n \times n$ matrix function defined by

$$Q_k(s) = A_0 Q_{k-1}(s) + A_1 Q_{k-1}(s-h) + A_2 Q_{k-1}(s-2h) + A_3 Q_{k-1}(s-3h) \text{ for } k = 1,2,3,\ldots s > 0,$$

with intial conditions:

$$Q_0(0) = I_n; Q_0(s) = 0; s \neq 0.$$

These initial conditions guarantee the unique solvability of the matrix function $Q_k(s)$

2.2 Preliminary Lemma on Determining matrices $Q_k(s), s \in \mathbf{R}$ (Ukwu & Temuru, 2018)

(i) $Q_k(s) = 0$ if $s < 0$

(ii) $Q_k(0) = A_0^k$

(iii) $Q_k(s) = 0$ if $s \neq rh$ for any integer r

(iv) $Q_k(h) = \sum_{(v_1 \dots v_k) \in P_{0(k-1),\,1(1)}} \prod_{j=1}^{k} A_{v_j}$; $k \geq 1$

(v) $Q_1(jh) = A_j \operatorname{sgn}(\max\{4-j, 0\})$

3. RESULTS AND INTERPRETATION

Main Result from (Ukwu & Temuru, 2018).

3.1 Theorem on the Functional Form of $Q_k(jh)$, for $j \geq 3k-3, k \geq 1$

$$Q_k(jh) = \begin{cases} 0 & \text{if } j \geq 3k+1 \quad \text{(i)} \\ A_3^k & \text{if } j = 3k \quad \text{(ii)} \\ \sum_{(v_1 \dots v_k) \in p_{2(1),\,3(k-1)}} \prod_{i=1}^{k} A_{v_i} = \sum_{r=0}^{k-1} A_3^r A_2 A_3^{k-1-r} & \text{if } j = 3k-1 \quad \text{(iii)} \\ \sum_{(v_1 \dots, v_k) \in p_{1(3k-1-j),\,3(j+1-2k)}} \prod_{i=1}^{k} A_{v_i} + \sum_{(v_1 \dots, v_k) \in p_{2(3k-j),\,3(j-2k)}} \prod_{i=1}^{k} A_{v_i} & \text{if } j = 3k-2 \quad \text{(iv)} \\ \sum_{(v_1 \dots v_k) \in p_{1(1),\,2(1),\,3(k-2)}} \prod_{i=1}^{k} A_{v_i} + \sum_{(v_1, \dots, v_k) \in p_{0(1),\,3(k-1)}} \prod_{i=1}^{k} A_{v_i} + \sum_{(v_1, \dots v_k) \in p_{2(3),\,3(k-3)}} \prod_{i=1}^{k} A_{v_i}, & j = 3k-3 \quad \text{(v)} \end{cases}$$

Note: For $j = 3k-3$, (v) can be also be expressed as:

$$Q_k(jh) = \sum_{r_0=0}^{1} \sum_{r_1=0}^{r_0+1} \sum_{(v_1 \cdots v_k) \in p_{0(r_0),1(r_1),2(3-3r_0-2r_1),3(k-3+2r_0+r_1)}} \prod_{j=1}^{k} A_{v_j}$$

The formulation and proof of the expressions for the determining matrices of the system of interest were achieved by the exploitation of key facts on permutations of objects, the interrogation of the feasibility dispositions of the determining matrices,the application of the principle of mathematical induction and the greatest integer function.

This article is a sequel to Ukwu and Temuru (2018) with further results and illumination on the subject of interest. Consider the system (1) and (2), with their standing hypotheses. Then

3.2 Theorem on $Q_k(jh)$; $1 \le k \le j$

$$Q_k(jh) = \begin{cases} \sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1 \cdots v_k) \in p_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1} \cdots A_{v_k} + \sum_{r_3=1}^{\left[\left[\frac{j}{3}\right]\right]} \sum_{r_0=0}^{\left[\left[\frac{3k-j}{3}\right]\right]} \sum_{(v_1,v_2,\dots v_k) \in p_{3(r_3),0(r_0),1(r_1),2(k-r_0-r_1-r_3)}} \prod_{i=1}^{k} A_{v_i}, 1 \le k \le j \le 5, & \text{(i)} \\ \sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1 \cdots v_k) \in p_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1} \cdots A_{v_k}, 0 \le j \le 2 & \text{(ii)} \\ 0, \ j \ge 3k+1, \ k \ge 1 & \text{(iii)} \end{cases}$$

where $r_1 = \max\{0, 3k - 2j + 3(r_3 - r_0)\}$ and $[[.]]$ denotes the greatest integer function.

3.3 Verification of Theorem 3.2

The mathematical convention of discarding infeasible components or equating them to zero, as applicable, is preserved here.

(ii) and (iii) follow respectively from the fact that

$$\left[\left[\frac{j}{3}\right]\right] = 0, \text{for } 0 \le j \le 2 \text{ and } \left[\left[\frac{3k-j}{3}\right]\right] < 0, \text{ for } j \ge 3k+1$$

(i) $k = 1, j \in \{1,2\} \Rightarrow \left[\left[\frac{j}{3}\right]\right] = 0 \Rightarrow Q_1(jh) = \sum_{r=0}^{\left[\left[\frac{2-j}{2}\right]\right]} \sum_{v_1 \in p_{0(r),1(2-j-2r),2(r+j-1)}} A_{v_1} \Rightarrow Q_1(h) = A_1, Q_1(2h) = A_2.$

$$k = 1, j \in \{3,4,\cdots\} \Rightarrow Q_1(jh) = \sum_{r_3=1}^{\left[\left[\frac{j}{3}\right]\right]} \sum_{r_0=0}^{\left[\left[\frac{3-j}{3}\right]\right]} \sum_{v_1 \in p_{3(r_3),0(r_0),1(r_1),2(1-r_0-r_1-r_3)}} A_{v_1} \Rightarrow Q_1(3h) = A_3, Q_1(jh) = 0, \ \forall j \ge 4.$$

These are consistent with Theorem 4.1 of (Ukwu & Temuru, 2018)

$k = 2, j = 2 \Rightarrow Q_2(2h) = \sum_{r=0}^{1} \sum_{(v_1,v_2) \in p_{0(r),1(2-2r),2(r)}} A_{v_1} A_{v_2} = A_1^2 + A_0 A_2 + A_2 A_0$, consistent with (Ukwu, 2014a) and direct evaluation from the determining equation.

$$k = 2, j = 3 \Rightarrow r = 0, r_3 = 1, r_0 \in \{0,1\}; r_0 = 0 \Rightarrow r_1 = 3 \Rightarrow k - r_0 - r_1 - r_3 = -2 < 0 \Rightarrow \text{infeasibility} \Rightarrow r_0 \ne 0$$

$$\Rightarrow r_0 = 1 \Rightarrow r_1 = 0 \Rightarrow Q_2(3h) = \sum_{(v_1,v_2) \in p_{0(0),1(1),2(1)}} A_{v_1} A_{v_2} + \sum_{(v_1,v_2) \in p_{3(1),0(1),2(0)}} \prod_{i=1}^{2} A_{v_i} = A_1 A_2 + A_2 A_1 + A_3 A_0 + A_0 A_3$$

This agrees with (v), theorem 4.2 of Ukwu and Temuru (2018).

$k=2, j=4 \Rightarrow r=0, r_3=1, r_0=0, r_1=1, r_2=0 \Rightarrow Q_k(4h)=A_2^2+A_1A_3+A_3A_1$, in agreement with (iv), theorem 4.2 of Ukwu and Temuru (2018).

$k=2, j=5 \Rightarrow r$ is infeasible, $r_3=1, r_0=0, r_1=0, r_2=1 \Rightarrow Q_k(5h)=A_2A_3+A_3A_2$, in agreement with (iii), theorem 4.2 of Ukwu and Temuru (2018).

$j=k=3 \Rightarrow r \in \{0,1\}, r_3=1, r_0 \in \{0,1,2\}; r_0=0 \Rightarrow r_1=6 \Rightarrow r_2=-4<0,$ resulting in summation infeasibility. $r_0=1 \Rightarrow r_1=\max\{0,9-6\}=3 \Rightarrow r_2<0,$ implying summation infeasibility.

$r_0=2 \Rightarrow r_1=\max\{0,9-6-3\}=0 \Rightarrow r_2=3-2-0-1=0,$ impling that

$$Q_3(3h)=A_0A_0A_3+A_0A_1A_2+A_0A_2A_1+A_0A_3A_0+A_1A_0A_2+A_1A_1^2+A_1A_2A_0+A_2A_0A_1+A_2A_1A_0+A_3A_0^2$$

This is consistent with the result from the determining equation:

$$\begin{aligned}Q_3(3h)&=A_0Q_2(3h)+A_1Q_2(2h)+A_2Q_2(h)+A_3Q_2(0)\\&=A_0(A_0A_3+A_1A_2+A_2A_1+A_3A_0)+A_1(A_0A_2+A_1^2+A_2A_0)+A_2(A_0A_1+A_1A_0)+A_3A_0^2\end{aligned}$$

$k=3, j=4, \Rightarrow r \in \{0,1\}, r_0 \in \{0,1\}, r_3=1; r_0=0 \Rightarrow r_1=4 \Rightarrow r_2<0, r_0=1 \Rightarrow r_1=1;$

$$\begin{aligned}Q_3(4h)&=A_0A_1A_3+A_0A_2^2+A_0A_3A_1+A_1A_0A_3+A_1A_1A_2+A_1A_2A_1+A_1A_3A_0+A_2A_0A_2\\&\quad+A_2A_1^2+A_2A_2A_0+A_3A_0A_1+A_3A_1A_0\ ;\end{aligned}$$

this is consistent with the result from the determining equation:

$$\begin{aligned}Q_3(4h)&=A_0Q_2(4h)+A_1Q_2(3h)+A_2Q_2(2h)+A_3Q_2(h)\\&=A_0(A_1A_3+A_2^2+A_3A_1)+A_1(A_0A_3+A_1A_2+A_2A_1+A_3A_0)\\&\quad+A_2(A_0A_2+A_1^2+A_2A_0)+A_3(A_0A_1+A_1A_0)\end{aligned}$$

$j=5, k=3 \Rightarrow r=0, r_3=1, r_0 \in \{0,1\}; r_0=0 \Rightarrow r_1=2 \Rightarrow r_2=0$

$$\begin{aligned}\Rightarrow Q_3(5h)&=A_0A_2A_3+A_0A_3A_2+A_1A_1A_3+A_1A_2^2+A_1A_3A_1+A_2A_0A_3+A_2A_1A_2+A_2A_2A_1\\&\quad+A_2A_3A_0+A_3A_0A_2+A_3A_1^2+A_3A_2A_0.\end{aligned}$$

This tallies with the following result from the determining equation:

$$\begin{aligned}Q_3(5h)&=A_0Q_2(5h)+A_1Q_2(4h)+A_2Q_2(3h)+A_3Q_2 2(h)\\&=A_0(A_2A_3+A_3A_2)+A_1(A_1A_3+A_2^2+A_3A_1)+A_2(A_0A_3+A_1A_2+A_2A_1+A_3A_0)\\&\quad+A_3(A_0A_2+A_1^2+A_2A_0)\end{aligned}$$

$j=4, k=4 \Rightarrow r \in \{0,1,2\}, r_0 \in \{0,1,2\}, r_3=1; r_0=0 \Rightarrow r_1=7 \Rightarrow$ the corresponding sum is infeasible
$r_0=1 \Rightarrow r_1=4 \Rightarrow r_2<0 \Rightarrow$ the corresponding sum is infeasible.

$r_0 = 2 \Rightarrow r_1 = 1 \Rightarrow r_2 = 0$. The feasible values: $r \in \{0,1,2\}, (r_0, r_1, r_2, r_3) = (2,1,0,1)$ yield

$$Q_4(4h) = A_0A_0A_1A_3 + A_0A_0A_2^2 + A_0A_0A_3A_1 + A_0A_1A_0A_3 + A_0A_1A_1A_2 + A_0A_1A_2A_1 + A_0A_1A_3A_0$$
$$+A_0A_2A_0A_2 + A_0A_2A_1^2 + A_0A_2^2A_0 + A_0A_3A_0A_1 + A_0A_3A_1A_0 + A_1A_0^2A_3 + A_1A_0A_1A_2 + A_1A_0A_2A_1$$
$$+A_1A_0A_3A_0 + A_1A_1A_0A_2 + A_1A_1^3 + A_1A_1A_2A_0 + A_1A_2A_0A_1 + A_1A_2A_1A_0 + A_1A_3A_0^2 + A_2A_0^2A_2$$
$$+A_2A_0A_1^2 + A_2A_0A_2A_0 + A_2A_1A_0A_1 + A_2A_1^2A_0 + A_2A_2^2A_0^2 + A_3A_0^2A_1 + A_3A_0A_1A_0 + A_3A_1A_0^2.$$

This coincides with the result from the determining equation:

$$Q_4(4h) = A_0Q_3(4h) + A_1Q_3(3h) + A_2Q_3(2h) + A_3Q_3(h)$$
$$= A_0\begin{pmatrix} A_0A_1A_3 + A_0A_2^2 + A_0A_3A_1 + A_1A_0A_3 + A_1A_1A_2 + A_1A_2A_1 + A_1A_3A_0 + A_2A_0A_2 \\ +A_2A_1^2 + A_2^2A_0 + A_3A_0A_1 + A_3A_1A_0 \end{pmatrix}$$
$$+ A_1\left(A_0^2A_3 + A_0A_1A_2 + A_0A_2A_1 + A_0A_3A_0 + A_1A_0A_2 + A_1^3 + A_1A_2A_0 + A_2A_0A_1 + A_2A_1A_0 + A_3A_0^2\right)$$
$$+A_2\left(A_0^2A_2 + A_0A_1^2 + A_0A_2A_0 + A_1A_0A_1 + A_1^2A_0 + A_2^2A_0^2\right) + A_3\left(A_0^2A_1 + A_0A_1A_0 + A_1A_0^2\right)$$

$j = 5, k = 4 \Rightarrow r \in \{0,1\}$, $r_0 \in \{0,1,2\}, r_3 = 1$; $r_0 = 0 \Rightarrow r_1 = 5 \Rightarrow$ the sum is infeasible.
$r_0 = 1 \Rightarrow r_1 = 2 \Rightarrow r_2 = 0$; $r_0 = 2 \Rightarrow r_1 = 0 \Rightarrow r_2 = 1$

The feasible values: $r \in \{0,1\}$; $(r_0, r_1, r_2, r_3) \in \{(1,2,0,1),(2,0,1,1)\}$ yield

$$Q_4(5h) = A_0A_0A_2A_3 + A_0A_0A_3A_2 + A_0A_1^2A_3 + A_0A_1A_2^2 + A_0A_1A_2A_1 + A_0A_1A_3A_1 + A_0A_2A_0A_3$$
$$+ A_0A_2A_1A_2 + A_0A_2^2A_1 + A_0A_2A_3A_0 + A_0A_3A_0A_2 + A_0A_3A_1^2 + A_0A_3A_2A_0 + A_1A_0A_1A_3 + A_1A_0A_2^2$$
$$+ A_1A_0A_3A_1 + A_1A_1A_0A_3 + A_1A_1^2A_2 + A_1A_1A_2A_1 + A_1A_1A_3A_0 + A_1A_2A_0A_2 + A_1A_2A_1^2 + A_1A_2^2A_0$$
$$+ A_1A_3A_0A_1 + A_1A_3A_1A_0 + A_2A_0^2A_3 + A_2A_0A_1A_2 + A_2A_0A_2A_1 + A_2A_0A_3A_0 + A_2A_1A_0A_2 + A_2A_1^3$$
$$+ A_2A_1A_2A_0 + A_2A_2A_0A_1 + A_2A_2A_1A_0 + A_2A_3A_0^2 + A_3A_0^2A_2 + A_3A_0A_1^2 + A_3A_0A_2A_0 + A_3A_1A_0A_1$$
$$+ A_3A_1^2A_0 + A_3A_2A_0^2,$$

coinciding with the result from the determining equation:

$$Q_4(5h) = A_0Q_3(5h) + A_1Q_3(4h) + A_2Q_3(3h) + A_3Q_3(2h)$$
$$= A_0\begin{pmatrix} A_0A_2A_3 + A_0A_3A_2 + A_1^2A_3 + A_1A_2^2 + A_1A_2A_1 + A_1A_3A_1 + A_2A_0A_3 + A_2A_1A_2 + A_2^2A_1 \\ +A_2A_3A_0 + A_3A_0A_2 + A_3A_1^2 + A_3A_2A_0 \end{pmatrix}$$
$$+ A_1\begin{pmatrix} A_0A_1A_3 + A_0A_2^2 + A_0A_3A_1 + A_1A_0A_3 + A_1^2A_2 + A_1A_2A_1 + A_1A_3A_0 + A_2A_0A_2 + A_2A_1^2 \\ +A_2^2A_0 + A_3A_0A_1 + A_3A_1A_0 \end{pmatrix}$$
$$+A_2\left(A_0^2A_3 + A_0A_1A_2 + A_0A_2A_1 + A_0A_3A_0 + A_1A_0A_2 + A_1^3 + A_1A_2A_0 + A_2A_0A_1 + A_2A_1A_0 + A_3A_0^2\right)$$
$$+ A_3\left(A_0^2A_2 + A_0A_1^2 + A_0A_2A_0 + A_1A_0A_1 + A_1^2A_0 + A_2A_0^2\right)$$

$j=5, k=5 \Rightarrow r \in \{0,1,2\}, r_0 \in \{0,1,2,3\}, r_3 = 1; r_0 = 0 \Rightarrow r_1 = 8 \Rightarrow$ the corresponding sum is infeasible

$r_0 = 1 \Rightarrow r_1 = \max\{0, 15-10+0\} = 5 > 5-1 \Rightarrow$ the sum is infeasible.

$r_0 = 2 \Rightarrow r_1 = 2 \Rightarrow r_2 = 5-2-2-1 = 0; r_0 = 3 \Rightarrow r_1 = 0 \Rightarrow r_2 = 1.$,

The feasible values: $r \in \{0,1,2,3\}$; $(r_0, r_1, r_2, r_3) \in \{(2,2,0,1),(3,0,1,1)\}$ yield

$$
\begin{aligned}
Q_5(5h) = {} & A_0A_0^2A_2A_3 + A_0A_0^2A_3A_2 + A_0A_0A_1^2A_3 + A_0A_0A_1A_2^2 + A_0A_0A_1A_3A_1 + A_0A_0A_2A_0A_3 \\
& + A_0A_0A_2A_1A_2 + A_0A_0A_2^2A_1 + A_0A_0A_2A_3A_0 + A_0A_0A_3A_0A_2 + A_0A_0A_3A_1^2 + A_0A_0A_3A_2A_0 \\
& + A_0A_1A_0A_1A_3 + A_0A_1A_0A_2^2 + A_0A_1A_0A_3A_1 + A_0A_1^2A_0A_3 + A_0A_1^3A_2 + A_0A_1^2A_2A_1 + A_0A_1^2A_3A_0 \\
& + A_0A_1A_2A_0A_2 + A_0A_1A_2A_1^2 + A_0A_1A_2^2A_0 + A_0A_1A_3A_0A_1 + A_0A_1A_3A_1A_0 + A_0A_2A_0^2A_3 + A_0A_2A_0A_1A_2 \\
& + A_0A_2A_0A_2A_1 + A_0A_2A_0A_3A_0 + A_0A_2A_1A_0A_2 + A_0A_2A_1^3 + A_0A_2A_1A_2A_0 + A_0A_2^2A_0A_1 + A_0A_2^2A_1A_0 \\
& + A_0A_2A_3A_0^2 + A_0A_3A_0^2A_2 + A_0A_3A_0A_1^2 + A_0A_3A_0A_2A_0 + A_0A_3A_1A_0A_1 + A_0A_3A_1^2A_0 + A_0A_3A_2A_0^2
\end{aligned}
$$

$$
\begin{aligned}
& +A_1A_0^2A_1A_3 + A_1A_0^2A_2^2 + A_1A_0^2A_3A_1 + A_1A_0A_1A_0A_3 + A_1A_0A_1^2A_2 + A_1A_0A_1A_2A_1 + A_1A_0A_1A_3A_0 \\
& +A_1A_0A_2A_0A_2 + A_1A_0A_2A_1^2 + A_1A_0A_2^2A_0 + A_1A_0A_3A_0A_1 + A_1A_0A_3A_1A_0 + A_1A_1A_0^2A_3 \\
& +A_1A_1A_0A_1A_2 + A_1A_1A_0A_2A_1 + A_1A_1A_0A_3A_0 + A_1A_1^2A_0A_2 + A_1A_1^4 + A_1A_1^2A_2A_0 + A_1A_1A_2A_0A_1 \\
& +A_1A_1A_2A_1A_0 + A_1A_1A_3A_0^2 + A_1A_2A_0^2A_2 + A_1A_2A_0A_1^2 + A_1A_2A_0A_2A_0 + A_1A_2A_1A_0A_1 + A_1A_2A_1^2A_0 \\
& +A_1A_2^2A_0^2 + A_1A_3A_0^2A_1 + A_1A_3A_0A_1A_0 + A_1A_3A_1A_0^2
\end{aligned}
$$

$$
\begin{aligned}
& A_2A_0^3A_3 + A_2A_0^2A_1A_2 + A_2A_0^2A_2A_1 + A_2A_0^2A_3A_0 + A_2A_0A_1A_0A_2 + A_2A_0A_1^3 + A_2A_0A_1A_2A_0 \\
& +A_2A_0A_2A_0A_1 + A_2A_0A_2A_1A_0 + A_2A_0A_3A_0^2 + A_2A_1A_0^2A_2 + A_2A_1A_0A_1^2 + A_2A_1A_0A_2A_0 \\
& +A_2A_1^2A_0A_1 + A_2A_1^3A_0 + A_2A_1A_2A_0^2 + A_2A_2A_0^2A_1 + A_2A_2A_0A_1A_0 + A_2A_2A_1A_0^2 + A_2A_3A_0^3 \\
& +A_3A_0^3A_2 + A_3A_0^2A_1^2 + A_3A_0^2A_2A_0 + A_3A_0A_1A_0A_1 + A_3A_0A_1^2A_0 + A_3A_0A_2A_0^2 + A_3A_1A_0^2A_1 \\
& +A_3A_1A_0A_1A_0 + A_3A_1^2A_0^2 + A_3A_2A_0^3.
\end{aligned}
$$

This is consistent with the result of the determining equation:

$$
Q_5(5h) = A_0Q_4(5h) + A_1Q_4(4h) + A_2Q_4(3h) + A_3Q_4(2h)
$$

$$
= A_0 \begin{pmatrix}
A_0^2A_2A_3 + A_0^2A_3A_2 + A_0A_1^2A_3 + A_0A_1A_2^2 + A_0A_1A_3A_1 + A_0A_2A_0A_3 + A_0A_2A_1A_2 \\
+A_0A_2^2A_1 + A_0A_2A_3A_0 + A_0A_3A_0A_2 + A_0A_3A_1^2 + A_0A_3A_2A_0 + A_1A_0A_1A_3 + A_1A_0A_2^2 \\
+A_1A_0A_3A_1 + A_1^2A_0A_3 + A_1^3A_2 + A_1^2A_2A_1 + A_1^2A_3A_0 + A_1A_2A_0A_2 + A_1A_2A_1^2 \\
+A_1A_2^2A_0 + A_1A_3A_0A_1 + A_1A_3A_1A_0 + A_2A_0^2A_3 + A_2A_0A_1A_2 + A_2A_0A_2A_1 + A_2A_0A_3A_0 \\
+A_2A_1A_0A_2 + A_2A_1^3 + A_2A_1A_2A_0 + A_2^2A_0A_1 + A_2^2A_1A_0 + A_2A_3A_0^2 + A_3A_0^2A_2 + A_3A_0A_1^2 \\
+A_3A_0A_2A_0 + A_3A_1A_0A_1 + A_3A_1^2A_0 + A_3A_2A_0^2
\end{pmatrix}
$$

$$
+A_1\begin{pmatrix} A_0^2A_1A_3+A_0^2A_2^2+A_0^2A_3A_1+A_0A_1A_0A_3+A_0A_1^2A_2+A_0A_1A_2A_1+A_0A_1A_3A_0+A_0A_2A_0A_2 \\ +A_0A_2A_1^2+A_0A_2^2A_0+A_0A_3A_0A_1+A_0A_3A_1A_0+A_1A_0^2A_3+A_1A_0A_1A_2+A_1A_0A_2A_1 \\ +A_1A_0A_3A_0+A_1^2A_0A_2+A_1^4+A_1^2A_2A_0+A_1A_2A_0A_1+A_1A_2A_1A_0+A_1A_3A_0^2+A_2A_0^2A_2 \\ +A_2A_0A_1^2+A_2A_0A_2A_0+A_2A_1A_0A_1+A_2A_1^2A_0+A_2^2A_0^2+A_3A_0^2A_1+A_3A_0A_1A_0+A_3A_1A_0^2 \end{pmatrix}
$$

$$
+A_2\begin{pmatrix} A_0^3A_3+A_0^2A_1A_2+A_0^2A_2A_1+A_0^2A_3A_0+A_0A_1A_0A_2+A_0A_1^3+A_0A_1A_2A_0+A_0A_2A_0A_1 \\ +A_0A_2A_1A_0+A_0A_3A_0^2+A_1A_0^2A_2+A_1A_0A_1^2+A_1A_0A_2A_0+A_1^2A_0A_1+A_1^3A_0+A_1A_2A_0^2 \\ +A_2A_0^2A_1+A_2A_0A_1A_0+A_2A_1A_0^2+A_3A_0^3 \end{pmatrix}
$$

$$
+A_3\begin{pmatrix} A_0^3A_2+A_0^2A_1^2+A_0^2A_2A_0+A_0A_1A_0A_1+A_0A_1^2A_0+A_0A_2A_0^2+A_1A_0^2A_1+A_1A_0A_1A_0 \\ +A_1^2A_0^2+A_2A_0^3 \end{pmatrix}
$$

3.4 Theorem on $Q_k\left(jh\right); 1\le j\le k\le 5$

For $1\le j\le k; j,k$ integers

$$
Q_k\left(jh\right)=\begin{cases} \displaystyle\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\ \sum_{(v_1\cdots v_k)\in p_{0\,(r+k-j),1(j-2r)\,,2(r)}} A_{v_1}\cdots A_{v_k}+\sum_{r_0=\text{lower}(j,k)}^{\text{Upper}(j,k)}\ \sum_{r_3=1}^{\left[\left[\frac{j}{3}\right]\right]}\ \sum_{(v_1\cdots v_k)\in p_{0\,(r_0),3(r_3)\,,0(r_2),\,1\,(r_1)}}\ \prod_{j=1}^{k}A_{v_j} \\ \displaystyle\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\ \sum_{(v_1\cdots v_k)\in p_{0\,(r+k-j),1(j-2r)\,,2(r)}} A_{v_1}\cdots A_{v_k},1\le j\le 2 \end{cases}
$$

where

$$
\text{lower}\left(j,k\right)=\begin{cases} k-1,\text{ if } j=3 \\ 2+\text{sgn}\left(\max\left\{0,k-j\right\}\right),\ j\ge 4 \end{cases}
$$

$$
\text{Upper}\left(j,k\right)=k-1-\text{sgn}\left(\max\left\{0,j-3\right\}\right)
$$

$$
r_2=\max\left\{k-j-1+3r_3-r_0-\text{sgn}\left(\max\left\{0,j-3\right\}\right)+3\text{sgn}\left(\max\left\{0,r_0-2-\left[\left[\frac{6-j}{2}\right]\right]\right\}\right)\right\}\text{sgn}\left(\left|j-3\right|\right)
$$

$$
r_1=k-\left(r_0+r_2+r_3\right).
$$

3.5 Proofof Theorem 3.4

$1\le k\le 2\Rightarrow 0\le j\le 2$, hencethe upper limit for r_3 is zero, which is less than the lower limit, implying infeasibility of the second summation component.

Therefore,

$$Q_k\left(jh\right)=\left\{\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{(v_1\cdots v_k)\in p_{0\,(r+k-j),1(j-2r)\,,2(r)}}A_{v_1}\cdots A_{v_k}\right.$$

The resulting $Q_k\left(jh\right)$'s are free of A_3.

Thus, we need only prove the formula for $3\le k\le 5$.

$k=3;\ j=3\Rightarrow r_3=1$, yielding lower $(3,3)=2$ and Upper$(3,3)=2\Rightarrow r_0=2$ and $r_2=0.$

$$Q_3\left(3h\right)=\sum_{r=0}^{1}\sum_{(v_1,v_2,v_3)\in p_{0\,(r),1(3-2r)\,,2(r)}}A_{v_1}A_{v_2}A_{v_3}+\sum_{(v_1,v_2,v_3)\in p_{0\,(2)\,,\,3(1),2(0),1(0)}}\prod_{i=1}^{3}A_{v_i}$$

$$=A_1^3+A_1A_2A_3+A_1A_3A_2+A_2A_1A_3+A_2A_3A_1+A_3A_1A_2+A_3A_2A_1+A_0^2A_3+A_0A_3A_0+A_3A_0^2,$$

as required.

$k=4;\ j=3\Rightarrow r\in\{0,1\},\ r_3=1, r_2=0,\text{lower}(3,4)=\text{Upper}(3,4)=3, r_0=3, r_1=0.$

,

Therefore,

$$Q_4\left(3h\right)=\sum_{r=0}^{1}\sum_{(v_1\cdots v_k)\in p_{0\,(r+1),1(3-2r)\,,2(r)}}A_{v_1}\cdots A_{v_k}+\sum_{(v_1,v_2,v_3,v_4)\in P_{0(3),3(1)}}\prod_{i=1}^{4}A_{v_i}$$

$$=A_0A_1^3+A_1A_0A_1^2+A_1^2A_0A_1+A_0A_1^3+A_0^2A_1A_2+A_0^2A_2A_1+A_0A_1A_2A_0+A_0A_2A_1A_0$$
$$+A_0A_2A_0A_1+A_0A_1A_0A_2+A_1A_0^2A_2+A_1A_2A_0^2+A_1A_0A_2A_0+A_2A_0^2A_1+A_2A_1A_0^2+A_2A_0A_1A_0$$
$$+A_0^3A_1+A_0A_1A_0^2+A_0^2A_1A_0+A_1A_0^3$$

This is in agreement with the following direct computation from the determining equation:

$$Q_4\left(3h\right)=A_0Q_3\left(3h\right)+A_1Q_3\left(2h\right)+A_2Q_3\left(h\right)+A_3Q_3\left(0\right)$$
$$=A_0\left(A_0A_1A_2+A_0A_2A_1+A_1A_0A_2+A_1^3+A_1A_2A_0+A_2A_0A_1+A_2A_1A_0+A_3A_0^2+A_0A_3A_0+A_0^2A_3\right)$$
$$+A_1\left(A_2A_0^2+A_0A_2A_0+A_0^2A_2++A_0A_1^2+A_1A_0A_1+A_1^2A_0\right)+A_2\left(A_1A_0^2+A_0A_1A_0+A_0^2A_1\right)+A_3A_0^3$$

$k=5;\ j=3\Rightarrow r\in\{0,1\}, r_3=1,$ yielding,

lower$(3,5)=4$ and Upper$(3,5)=4\Rightarrow r_0=4,$and $r_2=0, r_1=0.\Rightarrow r_0=4,$and $r_2=0\ \Rightarrow r_1=0$.

Therefore

$$Q_5\left(3h\right)=\left\{\sum_{r=0}^{1}\sum_{(v_1,v_2,v_3,v_4,v_5)\in p_{0\,(r+2),1(3-2r)\,,2(r)}}A_{v_1}\cdots A_{v_k}+\sum_{(v_1,v_2,v_3,v_4,v_5)\in P_{0(4),3(1)}}\prod_{i=1}^{5}A_{v_i}\right.$$

This coincides with the following result from the determining equation:

$$Q_5(3h) = A_0Q_4(3h) + A_1Q_4(2h) + A_2Q_4(h) + A_3Q_4(0)$$

$$= A_0\begin{pmatrix} A_0^3A_3 + A_0^2A_1A_2 + A_0^2A_2A_1 + A_0^2A_3A_0 + A_0A_1A_0A_2 + A_0A_1^3 + A_0A_1A_2A_0 + A_0A_2A_0A_1 \\ +A_0A_2A_1A_0 + A_0A_3A_0^2 + A_1A_0^2A_2 + A_1A_0A_1^2 + A_1A_0A_2A_0 + A_1^2A_0A_1 + A_1^3A_0 + A_1A_2A_0^2 \\ +A_2A_0^2A_1 + A_2A_0A_1A_0 + A_2A_1A_0^2 + A_3A_0^3 \end{pmatrix}$$

$$+ A_1\begin{pmatrix} A_0^3A_2 + A_0^2A_1^2 + A_0^2A_2A_0 + A_0A_1A_0A_1 + A_0A_1^2A_0 + A_0A_2A_0^2 + A_1A_0^2A_1 + A_1A_0A_1A_0 \\ +A_1^2A_0^2 + A_2A_0^3 \end{pmatrix}$$

$$+ A_2\left(A_0^3A_1 + A_0^2A_1A_0 + A_1A_0^2 + A_1A_0^3\right) + A_3A_0^4$$

$k = 4;\ j = 4 \Rightarrow r \in \{0,1,2\}$, $r_3 = 1$, yielding $\text{lower}(4,4) = 2$ and $\text{Upper}(4,4) = 2 \Rightarrow r_0 = 2$,

$$r_2 = \max\left\{4-4-1+3-2-1+3\operatorname{sgn}\left(\max\left\{0,2-2-\left[\left[\frac{6-4}{2}\right]\right]\right\}\right)\right\} = \max\{-1,0\} = 0 \Rightarrow r_1 = 1,$$

$$\Rightarrow Q_4(4h) = \sum_{r=0}^{2}\ \sum_{(v_1,\cdots,v_4)\in p_{0(r),1(4-2r),2(r)}} A_{v_1}\cdots A_{v_k} + \sum_{(v_1,v_2,v_3,v_4)\in P_{0(2),1(1)3(1)}}\ \prod_{i=1}^{4} A_{v_i},$$

in consistency with $Q_4(4h)$, for $1 \le k \le j \le 5$

$k = 5;\ j = 4 \Rightarrow r_3 = 1$, yielding $\text{lower}(4,5) = 3$ and $\text{Upper}(4,5) = 3 \Rightarrow r_0 = 3, r_1 = 1, r_2 = 0,$.

$$\Rightarrow Q_5(4h) = \left\{\sum_{r=0}^{2}\ \sum_{(v_1\cdots v_k)\in P_{0(r+1),1(4-2r),2(r)}} A_{v_1}\cdots A_{v_5} + \sum_{(v_1,v_2,v_3,v_4,v_5)\in P_{0(3),1(1)3(1)}}\ \prod_{i=1}^{5} A_{v_i}\right.$$

$k = 5;\ j = 5 \Rightarrow r_3 = 1$; yielding $\text{lower}(5,5) = 2$ and $\text{upper}(5,5) = 3$

and $\text{upper}(5,5) = 5-1-\operatorname{sgn}(\max\{0,5-3\}) = 3 \Rightarrow r_0 \in \{2,3\}$.

$$r_0 = 2 \Rightarrow r_2 = \max\left\{\begin{matrix} 5-5-1+3-2-\operatorname{sgn}(\max\{0,5-3\}) \\ +3\operatorname{sgn}\left(\max\left\{0,2-2-\left[\left[\frac{6-5}{2}\right]\right]\right\}\right),0 \end{matrix}\right\} = \max\{-1+3(0),0\} = 0 \Rightarrow r_1 = 2$$

$$r_0 = 3 \Rightarrow r_2 = \max\left\{\begin{matrix} 5-5-1+3-3-\operatorname{sgn}(\max\{0,5-3\}) \\ +3\operatorname{sgn}\left(\max\left\{0,3-2-\left[\left[\frac{6-5}{2}\right]\right]\right\}\right),0 \end{matrix}\right\} = \max\{-2+3(1),0\} = 1 \Rightarrow r_1 = 0$$

$$\Rightarrow Q_5(5h) = \left\{\begin{matrix} \displaystyle\sum_{r=0}^{2}\ \sum_{(v_1\cdots v_k)\in p_{0(r),1(5-2r),2(r)}} A_{v_1}\cdots A_{v_k} + \sum_{(v_1,v_2,v_3,v_4,v_5)\in P_{0(2),1(2)3(1)}}\ \prod_{i=1}^{5} A_{v_i} \\ \displaystyle +\sum_{(v_1,v_2,v_3,v_4,v_5)\in P_{0(3),2(1)3(1)}}\ \prod_{i=1}^{5} A_{v_i} \end{matrix}\right.$$

This agrees with $Q_5(5h)$, for $1 \le k \le j \le 5$, completing the proof of the theorem.

Theorems (3.2) and (3.4) can be robustly unified as follows:

3.6 Corollary to Theorems (3.2) and (3.4)

For $j, k \in \{0,1,2,3,4,5\}, j + k \neq 0$,

$$Q_k(jh) = \left(\sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1 \cdots v_k) \in p_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1} \cdots A_{v_k} + \sum_{r_3=1}^{\left[\left[\frac{j}{3}\right]\right]} \sum_{r_0=0}^{\left[\left[\frac{3k-j}{3}\right]\right]} \sum_{(v_1, v_2, \ldots v_k) \in p_{3(r_3),0(r_0),1(r_1),2(k-r_0-r_1-r_3)}} \prod_{i=1}^{k} A_{v_i} \right) \operatorname{sgn}\left(\max\{0, j+1-k\}\right)$$

$$+ \left(\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1 \cdots v_k) \in p_{0(r+k-j),1(j-2r),2(r)}} A_{v_1} \cdots A_{v_k} + \sum_{r_0=\text{lower}(j,k)}^{\text{Upper}(j,k)} \sum_{r_3=1}^{\left[\left[\frac{j}{3}\right]\right]} \sum_{(v_1 \cdots v_k) \in p_{0(r_0),3(r_3),0(r_2),1(r_1)}} \prod_{i=1}^{k} A_{v_i} \right) \operatorname{sgn}\left(\max\{0, k-j\}\right)$$

where
$$\text{lower}(j,k) = \begin{cases} k-1, \text{ if } j = 3 \\ 2 + \operatorname{sgn}\left(\max\{0, k-j\}\right), \ j \ge 4 \end{cases}$$

$$\text{Upper}(j,k) = k - 1 - \operatorname{sgn}\left(\max\{0, j-3\}\right)$$

$$r_2 = \max\left\{ \begin{array}{l} k - j - 1 + 3r_3 - r_0 - \operatorname{sgn}\left(\max\{0, j-3\}\right) + 3\operatorname{sgn}\left(\max\left\{0, r_0 - 2 - \left[\left[\frac{6-j}{2}\right]\right]\right\}\right) \\ +3\operatorname{sgn}\left(\max\left\{0, r_0 - 2 - \left[\left[\frac{6-j}{2}\right]\right]\right\}\right) \end{array} \right\} \operatorname{sgn}\left(|j-3|\right)$$

where $r_1 = \max\{0, 3k - 2j + 3(r_3 - r_0)\}$ in the second summation and $r_1 = k - (r_0 + r_2 + r_3)$ in the fourth..

Remark

Replacing $\operatorname{sgn}(\max\{0, j+1-k\})$ by $\operatorname{sgn}(\max\{0, j-k\})$ and $\operatorname{sgn}(\max\{0, k-j\})$ by $\operatorname{sgn}(\max\{0, k+1-j\})$ yields an equivalent functional form of $Q_k(jh)$.

3.7 Implications of Corollary 3.6

Corollary 3.6 immediately gives rise to the following controllability matrix and rank condition for the investigation of the Euclidean controllability of the initial function problem IFP (systems 1 and 2) on the interval $\left[0, \infty\right)$, provided $n \leq 6$.

The system (1) with (2) is Euclidean controllable on $\left[0, t_1\right]$ if and only if

$$\text{rank}\, \hat{Q}_n\left(t_1\right) = n,$$

where

$$\hat{Q}_n\left(t_1\right) = \left[Q_k\left(s\right)B : k \in \left\{0,1,\cdots,n-1; s \in \left\{0,h,\cdots,\left(\min\left\{(n-1),\left[\left[\left[\frac{t_1-h}{h}\right]\right]\right]\right\}\right)\right\}\right\}h \right],$$

$$\text{Dim}\, \hat{Q}_n\left(t_1\right) = n \times mn\left(\min\left\{n,\left[\left[\left[\frac{t_1}{h}\right]\right]\right]\right\}\right) = n \times mn\left(1+\min\left\{n-1,\left[\left[\left[\frac{t_1-h}{h}\right]\right]\right]\right\}\right)$$

where $[[[.]]]$ denotes the least integer function.

Therefore corollary 3.6 is a necessary and sufficient tool for the determination of the Euclidean controllability or otherwise of the IFP (systems 1 and 2).

4. SUMMARY AND CONCLUSION

This article obtained the structure of determining matrices $Q_k\left(s\right)$, of a class of triple-delay linear control systems, for $\max\left\{kh,s\right\} \leq 5h$. The obtained results can be deployed for the investigation of the Euclidean Controllability of triple-delay control modelson the global nonnegative interval,for state technology square matrices of order at most 6.The established structure of the controllability matrix obviates the need for the tedious step-wise computation of the associated determining matrices.Worthy of note is the fact that the obtained results have alleviated the computational constraints that have forced most authors to limit their computation of determining matrices even for much less complicated simple delay systems to the interval $\left[0,3h\right]$.

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