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POLYNOMIAL DIVISION VIA TEMPLATE MATRIX

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ABSTRACT

The division of a pair of giving polynomials to find its quotient and remainder is derived by applying the convolution matrix. The process of matrix formulation with a template matrix is simple, efficient and direct, comparing to the familiar classical longhand division and synthetic division. Typical numerical examples are provided to show the merit of the approaches.

KEYWORDS - Polynomial division; Longhand division; Synthetic division; Convolution matrix.

INTRODUCTION AND FORMULATION

The division of two univariate polynomials to find the quotient and remainder is expressed as

$$\frac{b(x)}{a(x)} = q(x) + \frac{r(x)}{a(x)}$$

or

$$b(x) = a(x)\,q(x) + r(x)$$

where $b(x)$ and $a(x)$ are the given dividend and divisor of degrees n and m, $n \geq m$, and $q(x)$ and $r(x)$ are the resulted quotient and remainder of degrees $n - m$ and $m - 1$ or less, respectively,

$$
\begin{aligned}
b(x) &= b_0 x^n + b_1 x^{n-1} + \cdots\cdots + b_{n-1} x + b_n \\
a(x) &= a_0 x^m + a_1 x^{m-1} + \cdots\cdots + a_{m-1} x + a_m
\end{aligned}
$$

$$
\begin{aligned}
q(x) &= q_0 x^{n-m} + q_1 x^{n-m-1} + \cdots\cdots + q_{n-m-1} x + q_{n-m} \\
r(x) &= r_0 x^{m-1} + r_1 x^{m-2} + \cdots\cdots + r_{m-2} x + r_{m-1}
\end{aligned}
$$

Then the coefficients of x^{ℓ} in both side of polynomial division equation after substituting of the expansion forms of $b(x)$, $a(x)$ and $q(x)$, $r(x)$ will give the following relation:

$$
b_{\ell} = a_{\ell} q_0 + a_{\ell-1} q_1 + \cdots\cdots + a_1 q_{\ell-1} + a_0 q_{\ell} + r_{\ell-(n-m+1)} , \qquad \ell = 0,1,\cdots\cdots,n
$$

where it is understood that

$$
b_{\ell} = 0,\ \ell > n, \quad a_{\ell} = 0,\ \ell > m, \quad \text{and} \quad q_{\ell} = 0,\ \ell > n-m, \quad r_{\ell} = 0,\ \ell < 0.
$$

The polynomial division is then written in the convolution matrix form [1,2] as,

$$
\begin{bmatrix} b_0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ b_n \end{bmatrix}
=
\begin{bmatrix}
a_0 & & & & 0 & & \\
\vdots & \ddots & & & & \ddots & \\
\vdots & & \ddots & & & & \ddots \\
\vdots & & & a_0 & & & 0 \\
\vdots & & & \vdots & 1 & & \\
a_m & & & \vdots & & \ddots & \\
 & \ddots & & \vdots & & & \ddots \\
 & & \ddots & \vdots & & & \ddots \\
 & & & a_m & & & 1
\end{bmatrix}
\begin{bmatrix} q_0 \\ \vdots \\ \vdots \\ q_{n-m} \\ r_0 \\ \vdots \\ \vdots \\ \vdots \\ r_{m-1} \end{bmatrix}
$$

The square matrix of this linear algebraic equation is non-singular, the unknown vector can thus be uniquely obtained, and must be computed recursively from top to bottom. It follows that the desired coefficients may also be computed by the following formulas:

$$
q_k = \Big(b_k - \sum_{\ell=\max(0,k-m)}^{k-1} a_{k-\ell}\, q_{\ell} \Big) \Big/ a_0 , \qquad k = 0,\cdots,n-m
$$

$$
r_{k-(n-m+1)} = \Big(b_k - \sum_{\ell=\max(0,k-m)}^{n-m} a_{k-\ell}\, q_{\ell} \Big), \qquad k = n-m+1,\cdots,n
$$

Applying the relationship among all of these entries, the polynomial division manipulation may further be caste into a template matrix M of order (n+1) x 4:

$$M = \begin{bmatrix} b_0 & a_0 & q_0 & \cdot \\ \vdots & \vdots & \vdots & \cdot \\ \vdots & \vdots & \vdots & \cdot \\ \vdots & \vdots & q_{n-m} & \cdot \\ \vdots & \vdots & \cdot & r_0 \\ \vdots & a_m & \cdot & \vdots \\ \vdots & \cdot & \cdot & \vdots \\ \vdots & \cdot & \cdot & \vdots \\ b_n & \cdot & \cdot & r_{m-1} \end{bmatrix}$$

A computer routine in MATLAB for the template matrix approach is presented. Inputs b and a, and outputs q and r are, respectively, the coefficient vectors of $b(x)$, $a(x)$, and $q(x)$ and $r(x)$. Also the template matrix is denoted as M.

```
function [q,r,M] = poly_div_M(b,a)
%
%    Polynomial division -- via template matrix M.
%    Given b(x) and a(x)  find q(x) and r(x)  in
%          b(x) = a(x)*q(x) + r(x).
%    Similar to MATLAB built-in function: 'deconv.m'.
%         F C Chang    09/18/18
%
  n = length(b)-1;  m = length(a)-1;
 if n < m,  q = 0;  r = b;   M = 0;   return, end;
  M = zeros(n+1,4);
  M(1:n+1,1) = b.';
  M(1:m+1,2) = a.';
for k = 1:n+1,  if k < n-m+2,
  M(k,3) = (M(k,1)-M(k:-1:1,2).'*M(1:k,3))/M(1,2);  else,
  M(k,4) =  M(k,1)-M(k:-1:1,2).'*M(1:k,3);          end;
end;
  q = M(1:n-m+1,3).';   r = M(n-m+2:n+1,4).';
```

TYPICAL EXAMPLES WITH REMARKS

For given

$$b(x) = 4x^8 + 5x^7 - x^6 + 7x^5 - 6x^4 + x^3 + 2x^2 - 3x + 7$$

$$a(x) = 3x^5 + x^4 - 7x^3 + 5x^2 - 4x + 2$$

in

$$b(x) = a(x)\,q(x) + r(x)$$

we shall find the desired results as

$$q(x) = \frac{4}{3}x^3 + \frac{11}{9}x^2 + \frac{64}{27}x \;+ \frac{176}{81}$$

$$r(x) = \frac{619}{81}x^4 + \frac{533}{81}x^3 - \frac{148}{81}x^2 + \frac{77}{81}x \;+ \frac{215}{81}$$

by applying either one of the following approaches:

(1) Longhand polynomial division

$$\begin{array}{r|rrrrrrrrr}
 & +\frac{4}{3} & +\frac{11}{9} & +\frac{64}{27} & +\frac{176}{81} & & & & & \\
\hline
3+1-7+5-4+2\;) & +4 & +5 & -1 & +7 & -6 & +1 & +2 & -3 & +7 \\
 & +4 & +\frac{4}{3} & -\frac{28}{3} & +\frac{20}{3} & -\frac{16}{3} & +\frac{8}{3} & & & \\
\cline{3-10}
 & & +\frac{11}{3} & +\frac{25}{3} & +\frac{1}{3} & -\frac{2}{3} & -\frac{5}{3} & +2 & -3 & +7 \\
 & & +\frac{11}{3} & +\frac{11}{9} & -\frac{77}{9} & +\frac{55}{9} & -\frac{44}{9} & +\frac{22}{9} & & \\
\cline{4-10}
 & & & +\frac{64}{9} & +\frac{80}{9} & -\frac{61}{9} & +\frac{29}{9} & -\frac{4}{9} & -3 & +7 \\
 & & & +\frac{64}{9} & +\frac{64}{27} & -\frac{448}{27} & +\frac{320}{27} & -\frac{256}{27} & +\frac{128}{27} & \\
\cline{5-10}
 & & & & +\frac{176}{27} & +\frac{265}{27} & -\frac{233}{27} & +\frac{244}{27} & -\frac{209}{27} & +7 \\
 & & & & +\frac{176}{27} & +\frac{176}{81} & -\frac{1232}{81} & +\frac{880}{81} & -\frac{704}{81} & +\frac{352}{81} \\
\cline{6-10}
 & & & & & +\frac{619}{81} & +\frac{533}{81} & -\frac{148}{81} & +\frac{77}{81} & +\frac{215}{81}
\end{array}$$

(2) Synthetic polynomial division

$$\begin{array}{r|rrrr:rrrrr}
 & +4 & +5 & -1 & +7 & -6 & +1 & +2 & -3 & +7 \\
-\frac{1}{3} & & -\frac{4}{3} & -\frac{11}{9} & -\frac{64}{27} & -\frac{176}{81} & & & & \\
+\frac{7}{3} & & & +\frac{28}{3} & +\frac{77}{9} & +\frac{448}{27} & +\frac{1232}{81} & & & \\
-\frac{5}{3} & & & & -\frac{20}{3} & -\frac{55}{9} & -\frac{320}{27} & -\frac{880}{81} & & \\
+\frac{4}{3} & & & & & +\frac{16}{3} & +\frac{44}{9} & +\frac{256}{27} & +\frac{704}{81} & \\
-\frac{2}{3} & & & & & & -\frac{8}{3} & -\frac{22}{9} & -\frac{128}{27} & -\frac{352}{81} \\
\hline
 & +4 & +\frac{11}{3} & +\frac{64}{9} & +\frac{176}{27} & +\frac{619}{81} & +\frac{533}{81} & -\frac{148}{81} & +\frac{77}{81} & +\frac{215}{81}
\end{array}$$

(3) Convolution polynomial division

$$\begin{bmatrix} +4 \\ +5 \\ -1 \\ +7 \\ -6 \\ +1 \\ +2 \\ -3 \\ +7 \end{bmatrix} = \left[\begin{array}{cccc|ccccc} +3 & & & & & & & & \\ +1 & +3 & & & & & & & \\ -7 & +1 & +3 & & & & & & \\ +5 & -7 & +1 & +3 & & & & & \\ \hline -4 & +5 & -7 & +1 & 1 & & & & \\ +2 & -4 & +5 & -7 & & 1 & & & \\ & +2 & -4 & +5 & & & 1 & & \\ & & +2 & -4 & & & & 1 & \\ & & & +2 & & & & & 1 \end{array}\right] \begin{bmatrix} +\frac{4}{3} \\ +\frac{11}{9} \\ +\frac{64}{27} \\ +\frac{176}{81} \\ +\frac{619}{81} \\ +\frac{533}{81} \\ -\frac{148}{81} \\ +\frac{77}{81} \\ +\frac{215}{81} \end{bmatrix}$$

(4) Polynomial division template matrix --- after run [q,r,M] = polydivM(b,a) on MATLAB.

$$M = \begin{bmatrix} +4 & +3 & +\frac{4}{3} & 0 \\ +5 & +1 & +\frac{11}{9} & 0 \\ -1 & -7 & +\frac{64}{27} & 0 \\ +7 & +5 & +\frac{176}{81} & 0 \\ -6 & -4 & 0 & +\frac{619}{81} \\ +1 & +2 & 0 & +\frac{533}{81} \\ +2 & 0 & 0 & -\frac{148}{81} \\ -3 & 0 & 0 & +\frac{77}{81} \\ +7 & 0 & 0 & +\frac{215}{81} \end{bmatrix}$$

Comparing of the approaches in this typical example, we found that the polynomial division template approach is much simple and effective. The desired quotient and remainder are readily computed without calculating any intermediate data as those in the familiar classical longhand polynomial division and synthetic polynomial division [3,4].

REFENCES

1 B. Dayton, "Polynomial GCDs by linear algebra," Theory of Equations, Northeastern Illinois University, www.neiu.edu/~bhdayton, 2004.

2 F. C. Chang, "Polynomial division by convolution," Applied Mathematics E-Notes, 11, 2011, 249-254.

3 L. Zhou, “Short division of polynomials,” The college Mathematics Journal, 40**,** 2009, 44-46.

4 D. Bini and V. Pan, “Polynomial division and its computational complexity,” Journal of Complexity, 2 1986, 179-203.