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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE BOUNDARY PROBLEM OF THE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS OF SCHRÖDINGER TYPE IN THE CASE OF COMPACT SUPPORT POTENTIALS

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ABSTRACT

The existence and uniqueness of Jost's Matrix Solutions in the compact case is showed using the Theory of Systems of Differential Equations. These solutions solve the boundary problem for the Schrödinger equation in one variable. Using unity and existence and compactness, it is possible to reduce the bounded problem into one of initial condition.

KEYWORDS – Boundary Problem, Initial Value Problem, Jost Matrix Solutions, Volterra Equations, Theory of ODES

INTRODUCTION

We shall deal with a boundary problem of a differential equation of order two that includes the existence and uniqueness of Jost's solutions to the boundary problem of the Schrödinger equation in the case of compact support. This problem has been solved by transforming the differential equation into an integral equation (Volterra's integral). The theory of ordinary differential equations (see e.g. [1]) solves initial condition problems by transforming such problems into an integral equation, which is not explicit in the results of the said theory of existence and uniqueness. This is notorious in second order equations greater than 1, where at least two values are needed to establish the uniqueness: the value of the solution and its derivative at a given point. The Strurm-Liouville theory deals with boundary problems at the extremes of a finite interval. In our case, the working interval of the considered system of equations is not bounded, but we can reduce it to an initial condition problem of a system of ordinary differential equations in a special case, such as when the support of the equation is compact.

SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF ORDER TWO

A system of equations

$$\begin{aligned} {y_1}'' &= f_1(x,y,y') \\ &\vdots \\ {y_n}'' &= f_n(x,y,y') \end{aligned}$$

has the matrix representation $Y'' = F(x.Y,Y')$, where Y and F are in general matrix valued functions such that de products and sum of matrices can be performed . A prime indicates the derivative of the primed function, for instance Y' denote s the derivative of the matrix function unknown Y .

Proposition. Let $Y" = F(x,Y,Y'), x \in I$, where I is an interval not bounded for the left or for the right, be the matrix representation of a system of differential equations of order two for which there is a unique solution to any initial condition given in I, and $F(x,Y,Y') = -a^2Y$ for $x \leq -\frac{1}{\varepsilon}$ for some ε then there exists a unique solution ψ_- tal que $e^{iax}\psi_- = \mathbf{I}$, $x \leq -\frac{1}{\varepsilon}$ where **I** is the identity matrix of suitable size. Similarly, there is ψ_+such that $e^{-iax}\psi_+ = \mathbf{I}$, provided that $F(x,Y,Y') = -a^2Y$ for $x \geq \frac{1}{\varepsilon}$.

Proof.. Assume tha de interval is no bounded by the left. For the hypothesis of the existence of the solution to any initial condition for the given equation, there is a unique solution ψ_- such that $(\psi_-)^{(j)}(x_0) = (-ia)^j e^{iax_0}\mathrm{I}$, $x_0 = -\varepsilon^{-1}, j = 0,1$, where I is the identity matrix. For $x \leq x_0$, ϕ_- is also the solution of the differential equation of order two with constant coefficients $Y'' + a^2Y = 0$, with the same initial condition, whose solution is $e^{-iax}\mathrm{I}$. From uniqueness of the solution in the interval $x \geq \mathrm{x}_0$, we have $\psi_-(x) = e^{-iax}\mathrm{I}, \mathrm{x} \leq \mathrm{x}_{0,\text{which is unque by hipothesis}}$. Similarly, in the case that the interval is non bounded by the right, there existe a unique solution such that $\psi_+ \; defined \; in \; the \; interval \;\; I$

That satisfies that $e^{-iax}\psi_+ = \mathbf{I}$ for $x \geq x_0 = \varepsilon^{-1}$.

Remark. In the case that $\mathrm{Im}a > 0$ then $\psi_\pm$ decay exponentially. In the case that $\mathrm{Im}a < 0$ there are others two solutions that decay exponentially, whose proof of existence and uniqueness is similar to those presented in the preceding Propositionn.

Theorem. The system of linear ODES whose matrix representation is $-Y'' + VY = k^2Y$, where V is a continuous function with matrix values of compact support has two solutions $\psi_{\pm}$ such that $e^{\mp ikx}\psi_{\pm} = \mathrm{I}, x \geq b, x \leq a,$ respectively where a, b are the upper and lower extremes of the interval I , respectively. If $\mathrm{Im}k > 0$ both solutions decay exponentially.
Proof. The linearity of the system, continuity of the function V and that its support is compact shows the existence of a unique solution in $(-\infty, +\infty)$ for any given initial condition in view of the Theory of linear systems of differential equations [1], since in this case the Lipchitz constant is independent of the variables x, Y, Y'. Outside of the support of V . the equation is $Y'' + k^2Y = 0$, so we have the hypotheses of the previous proposition, reason why we can conclude the existence of the functions $\psi_{\pm}$ such that $\psi_{-}e^{iak} = I, x \leq a = \varepsilon^{-1}\mathrm{sgn}a, \varepsilon = |a|^{-1}$ y $\psi_{+}e^{ikx} = I, x \geq b = \varepsilon^{-1}sgnb, \varepsilon = |b|^{-1}$. From the definition of such functions, it was that they decay exponentially if $\mathrm{Im}k > 0$. Here sgn means the sign of the given number.

Remarks

a) When V is a function with scalar values, the equation takes the name of Schrödinger's equation with potential V and its solutions are called Jost's solutions. In the general case, when V is a function of matrix values then the system of equations whose matrix representation is said differential matrix equation, are called ODES systems of the Schrödinger type or simply Matrix Schrödinger equation and to the matrix solutions as Jost matrix solutions.

b) We have that the solutions $\psi_{\pm}$ satisfy the boundary conditions $\lim_{x\to\pm\infty} e^{iak}Y = \boldsymbol{I}$, respectively.

c) Using Green's functions and Fourier transform e.g. [2], for each boundary condition, the Schrödinger matrix equation is transformed into a Volterra-type integal matrix equation, so from the previous Theorem, we have that ψ_{-} and ψ_{+} $satisfy$ the Volterra integral equtios

$$e^{ikx}\psi_{-}(x) = I + \int_{-\infty}^{x} \frac{1-\exp\left(2ik(x-y)\right)}{2ik} V(y)e^{iky}\psi_{-}(y)dy \text{ y}$$
$$e^{-ikx}\psi_{+}(x) = I + \int_{x}^{+\infty} \frac{1-\exp-2ik(x-y)}{2ik} V(y)e^{-iky}\psi_{+}(y)dy, \mathrm{Im}k > 0., \text{ respectively.}$$

d) If $x < a$, then the integral of the first equation is zero and if $x > b$ the integral in the second equation is also zero, then the boundary conditions are confirmed immediately.

CONCLUSIONS

The Jost's matrix solutions can be obtained directly from the Theory of Ordinary Differential Linear Equations. Traditionally they have been obtained from the Theory of Integral Equations of the Volterra type [3], since when formulated in this way the boundary condition is considered because the integrals are defined in the integral formulation. The theory of linear systems of differential equations shows the existence and uniqueness of initial condition problems, but in our case, it is also possible to use it for boundary problems such as the existence of Jost Matrix functions, solutions to a boundary problem of a system of linear equations of differential equations of order two.

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