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COMMON FIXED POINT THEOREMS IN COMPLEX VALUED METRIC SPACE USING IMPLICIT RELATION

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Abstract:

Banach contraction principle in gives appropriate and simple conditions to establish the existence and uniqueness of a solution of an operator equation $Tx = x$. Later, a number of papers were devoted to the improvement and generalization of that result. Most of these results deal with the generalizations of the different contractive conditions in metric spaces. The aim of this paper is to prove the existence and uniqueness of a common fixed point for a pair of mappings satisfying occasionally weakly compatible maps in complex valued metric space using implicit relations. The obtained results generalize and extend some of the well-known results in the literature.

Keywords: Complex metric space, weakly compatible, occasionally weakly compatible, implicit relation.

Introduction:

Azam et al.[1] introduced the notion of complex valued metric spaces and established some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. Though complex valued metric spaces from a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. Indeed the definition of a cone metric space banks on the underlying Banach space which is not a division ring. However, in complex valued metric spaces, one can study improvements of a host of result of analysis involving division. One can refer related results in [3, 7]. Jungck generalized the concept of weak commuting mapping given by Sessa [12], by introducing the concept of compatible mapping in different. Many authors [2, 5, 6] proved fixed point theorems for compatible mappings in different types.

In this paper, we introduced some new common fixed point theorems for generalized contractive maps in complex-valued metric space by using these new properties.

For the sake of completeness, we recall some definitions and known results in complex valued metric space.

BASIC DEFINATIONS AND PRELIMINARIES

An ordinary metric d is a real- valued function from a set $X \times X$ into R. where X is a non-empty set. That is $\rho : X \times X \to R$. A Complex number $z \in C$ is an ordered pair of real number, whose first co-ordinate is called, Re(z) and second co-ordinate is Im(z). Thus a complex- valued metric d would be a function from a set $X \times X$ into C, where X is a non-empty set and C is the set of complex number. That is $\rho: X \times X \to R$.

Suppose $\mathbb{C}$ be the set of complex numbers throughout this section and $z_1, z_2 \in C$, recall a natural partial order relation $\preccurlyeq$ on C as follows: $z_1 \preccurlyeq z_2$ if and only if $Re(z_1) \le Re(z_2)$ and $Im(z_1) \le Im(z_2)$,Consequently, one can infer that $z_1 \preccurlyeq z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < (z_2)$

(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = (z_2)$

(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < (z_2)$

(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = (z_2)$

In particular, we write z_1 not $\preccurlyeq z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that $0 \preccurlyeq z_1$ not $\preccurlyeq z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \preccurlyeq z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 2.1. [1]. Let X be a nonempty set, whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d: X \times X \to \mathbb{C}$ satisfies thefollowing conditions:

(C_1) $0 \preccurlyeq \rho(x, y)$ for all x,y ∈X and $\rho(x,y) = 0$ if and only if $x = y$;

(C_2) $\rho(x, y) = \rho(y, x)$ for all x, y ∈ X;

(C_3) $\rho(x,y) \preccurlyeq \rho(x,z) + \rho(z,y)$ for all x,y,z ∈ X.

Then ρ is called a complex – valued metric on X, and (X,ρ) is called a complex- valued metric space.

Example 2.1. Define complex valued metric $\rho: X \times X \to C$ by $\rho(z_1, z_2) = e^{3i}|z_1, z_2|$.Then (X, ρ) is a complex valued metric space.

Definition 2.2.[1]Let (X, ρ) be a complex valued metric space and $B \subseteq X$.

(i) b ∈B is called an interior point of a set B whenever there is $0 \prec r \in \mathbb{C}$ such that $N(b, r) \subseteq B$,where $N(b, r) = \{y \in X: \rho(b, y) \prec r\}$.

(ii) A point x ∈X is called a limit point of B whenever for every $0 \prec r \in \mathbb{C}$, $N(x, r) \cap (B \setminus \{X\}) \neq \emptyset$.

(iii) A subset A ⊆ X is called open whenever each element of A is an interior point of A whereas a subset B⊆ X is called closed whenever each limit point of B belongs to B. The family F = {N(x, r): x∈ X,0 ≺ r} is a sub-basis for a topology on X. We denote this complex topology by τ_C. Indeed, thetopology τ_C is Hausdorff.

Definition 2.3.[1]Let (X, d) complex- valued metric space and x∈ X. Then sequence $\{x_n\}$ in X is

(i) Convergent if $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$, if for every 0≺ c ∈ C, there is a natural number N such that $\rho(x_n,x) \prec c$, for all n > N.We denote it by $lim_{n\to\infty} x_n = x$.

(ii) a Cauchy sequence, if for every c ∈ C, with 0 ≺ c there is a natural number N such that $\rho(x_n, x_m) \prec c$, for all n, m > N.

(iii) The metric space (X, ρ) is a complete complex valued metric space if every Cauchy sequence is convergent.

In a metric space, every convergent sequence is a Cauchy sequence but the converse is not true. For instance, Euclidean n-space with the Euclidean distance is complete metric space where as the set of rational numbers with metric $\rho(x,y) = |x - y|$ is not a complete metric space.

In 1968, Jungck [8] defined the concept of compatible mappings which is more general than that of commuting and weakly commuting mappings.

Definition 2..4.[9]A pair (f, g) of self-mappings of a metric space (X, ρ) into itself, is called compatible mapping if $lim_{n\to\infty}\rho(fgx_n, gfx_n) = 0$) whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty}fx_n = lim_{n\to\infty}gx_n = z$ for some z∈X.

Example 2.2 Let X = [0, ∞) be endowed with usual metric d and $f, g : X \to X$ such that $fx = x^3$ and $gx = 2x^3$. Then $fgx \neq gfx$. So, f and g are not commuting on X and $|fgx - gfx| > |fx - gx|$. Therefore, f and g are not weakly commuting on X. Also, for any sequence $\{x_n\}$ in X such that $lim_{n\to\infty}fx_n = lim_{n\to\infty}gx_n = u \in X$ then $lim_{n\to\infty}\ \rho(fgx_n, gfx_n) = lim_{n\to\infty}|fgx_n - gfx_n| = 0$.Therfore, f and g are compatible.

In 1998, Jungck and Rhoades [10] introduced the notion of weakly compatible mappings which is more general than that of compatibility as follows:

Definition 2.5.[10]A pair (f, g) of self-mappings of a metric space (X, ρ) into itself, is called weakly compatible mapping if they commute at all of their coincidence point i.e. fx = gx for some x ∈X implies fgx = gfx. Also, compatible mapping are weakly compatible but converse is not .true.

Example 2.3. Define complex –valued metricρ: X×X→C by such that ρ (z_1, z_2) =$e^{ia}|z_1-z_2|$ where a is any real constant. Then(X,ρ) is a complex valued metric space .Suppose self-maps A and S be defined as:

Az =$2e^{i\pi/4}$ if Re(z) ≠ 0, Az = $3e^{i\pi/3}$ if Re(z) = 0, and

Sz = $2e^{i\pi/4}$ if Re(z) ≠ 0, Sz= $4e^{i\pi/3}$ if Re(z)=0,

Then maps A and S are weakly compatible at all z ∈ C with Re (z) ≠0.

In 2008, Al Thagafi and Shahzad [2] introduced the concept of occasionally weakly compatible (owc) mappings which is a proper generalization of weakly compatible mappings.

Definition 2.6.[2]. Two self mappings f and g of a complex –valued metric space (X,ρ) are said to be occasionally weakly compatible (owc) if there is a point x in X which is a coincidence point of f and g at which f and g commute.

Example 2.4. Let X= [0,∞) with usual metric . define f,g:X→X by fx = 2x and gx = x^2, for all x ∈ X. Then fx = gx at x=0, 2 but fg (0) = gf(0) and fg(2) ≠ gf(2). Therefore, mappings f and g are occasionally weakly compatible but not weakly compatible.

Definition 2.7.[11]A pair (f, g) of self – mappings of a metric space (X, ρ) is said to be satisfy property (E.A), if here exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$= z for some z ∈ X.

Example 2.5. Let X=[0,∞).Define f,g:X→X by fx=$\frac{2x}{4}$ and gx= $\frac{5x}{4}$, for all x ∈ X. Consider the sequence $\{x_n\}$ = $\frac{2}{n}$ clearly, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$= 0 ∈X. Then f and g satisfy property (E.A).

Example 2.6. Let X = C and d be any complex valued metric .Define self maps A and S by Az =1-z and Sz = 1+z, for all z∈ X. Consider a sequence in X as $\{x_n\}$ = {1/n} where n =1, 2, 3, … then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$= 0.

Hence the pair (A, S) satisfies property (E.A) for the sequences $\{x_n\}$ in X.

Definition 2.8.[11]. Two pairs of self-maps (A, S) and (B, T) on a complex valued metric space (X,ρ) Satisfies common property (E.A) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n$= p for some p ∈ X.

Definition 2.1.9.[11]. Two finite families of self maps $\{A_i\}_{i=1}^{m}$ and $\{B_j\}_{j=1}^{m}$ on a set X are pair wise commuting if

(i) $A_iA_j = A_jA_i, i, j \in \{1,2,3, \dots m\}$,

(ii) $B_iB_j = B_jA_i, i, j \in \{1,2,3, \dots n\}$,

Implicit relations play important role in establishing of common fixed point results.

Let M_6 be the set of all continuous functions satisfying the following conditions:

(A) $\emptyset(u, 0, u, 0,0, u) \preccurlyeq 0 \Rightarrow u \preccurlyeq 0$

(B) $\emptyset(u, 0,0, u, u, 0) \preccurlyeq 0 \Rightarrow u \preccurlyeq 0$

(C) $\emptyset(u, u, 0,0, u, u) \preccurlyeq 0 \Rightarrow u \preccurlyeq 0$ for all $0 \preccurlyeq u$.

Example 3.1. Define $\emptyset$: $(C)^6 \rightarrow C$ as $\emptyset$ (t_1, t_2, t_3, t_4, t_5, t_6) = $t_1 - \emptyset_1$ (min {t_2, t_3, t_4,t_5,t_6}), where φ_1: C→C is increasing and continuous function such that $\emptyset_1(s) > s$ for all s∈C, clearly,$\emptyset$ satisfies all conditions (A), (B) and (C).Therefore, $\emptyset \in M_6$.

Our main theorem runs as follows.

Theorem 3.1.1. Let A, B, S, T, P and Q be six self mappings of a complex-valued metric space (X, ρ) satisfying the following conditions:

(i) $P(X) \subseteq AB(X), Q(X) \subseteq ST(X)$,

(ii) The pair (P, AB) and (Q, ST) share the common (E.A) property.

(iii) For any x, y ∈X,$\emptyset$ in M_6.

$$\emptyset \begin{Bmatrix} \rho(\mathrm{Px}, \mathrm{Qy}), \rho(\mathrm{ABx}, \mathrm{STy}), \rho(\mathrm{ABx}, \mathrm{Qy}), \\ \rho(\mathrm{STy}, \mathrm{Px}), \rho(\mathrm{ABx}, \mathrm{Px}), \rho(\mathrm{STy}, \mathrm{Qy}) \end{Bmatrix} \preccurlyeq 0$$

(iv) AB = BA, ST = TS, PB = BP, SQ = QS, QT = TQ.

Then the pair (P, AB) and (Q, ST) have a point of coincidence each. Moreover A, B, S, T, P and Q have a unique common fixed point provided both the pairs (P, AB) and (Q, ST) are occasionally weakly compatible.

Proof. In view of (ii), there exist two sequences $\{x_n\}$ and $\{y_n\}$in X such that

$lim_{n\rightarrow\infty}\mathrm{P}x_n = lim_{n\rightarrow\infty}\mathrm{AB}x_n = lim_{n\rightarrow\infty}\mathrm{B}y_n = lim_{n\rightarrow\infty}\mathrm{ST}y_n = z$ for some z ∈X.

since P(X) ⊂ AB(X), there exist a point u ∈X such that ABu = z.

Put $x = u$ and $y = y_n$ in (iii), we have

$$\emptyset \begin{Bmatrix} \rho(\mathrm{Pu}, \mathrm{Q}y_n), \rho(\mathrm{ABu}, \mathrm{ST}y_n), \rho(\mathrm{ABu}, \mathrm{Q}y_n), \\ \rho(\mathrm{ST}y_n, \mathrm{Pu}), \rho(\mathrm{ABu}, \mathrm{Pu}), \rho(\mathrm{ST}y_n, \mathrm{Q}y_n) \end{Bmatrix} \preccurlyeq 0$$

$$\emptyset \begin{Bmatrix} \rho(\mathrm{Pu}, \mathrm{z}), \rho(\mathrm{z}, \mathrm{z}), \rho(\mathrm{z}, \mathrm{z}), \\ \rho(\mathrm{z}, \mathrm{Pu}), \rho(\mathrm{z}, \mathrm{Pu}), \rho(\mathrm{z}, \mathrm{z}) \end{Bmatrix} \preccurlyeq 0$$

$$\emptyset \begin{Bmatrix} \rho(\mathrm{Pu}, \mathrm{z}), 0,0, \\ \rho(\mathrm{z}, \mathrm{Pu}), \rho(\mathrm{z}, \mathrm{Pu}), 0 \end{Bmatrix} \preccurlyeq 0$$

Using implicit relation (B), we get

$$\rho(\mathrm{Pu},\mathrm{z}) \preccurlyeq 0.$$

This gives Pu = z. Therefore Pu = Abu = z.

Since since Q(X) ⊂ ST(X), there exist a point v ∈ X such that STv = z.

Put x $= u$ and y $= v$ in (iii), we have

$$\emptyset\begin{Bmatrix}\rho(\mathrm{Pu},\mathrm{Q}v),\rho(\mathrm{ABu},\mathrm{ST}v),\rho(\mathrm{ABu},\mathrm{Q}v),\\ \rho(\mathrm{ST}v,\mathrm{Pu}),\rho(\mathrm{ABu},\mathrm{Pu}),\rho(\mathrm{ST}v,\mathrm{Q}v)\end{Bmatrix} \preccurlyeq 0$$

$$\emptyset\begin{Bmatrix}\rho(\mathrm{z},\mathrm{Q}v),\rho(\mathrm{z},\mathrm{z}),\rho(\mathrm{z},\mathrm{Q}v),\\ \rho(\mathrm{z},\mathrm{z}),\rho(\mathrm{z},\mathrm{z}),\rho(\mathrm{z},\mathrm{Q}v)\end{Bmatrix} \preccurlyeq 0$$

$$\emptyset\begin{Bmatrix}\rho(\mathrm{z},\mathrm{Q}v),0,\rho(\mathrm{z},\mathrm{Q}v),\\ 0,0,\rho(\mathrm{z},\mathrm{Q}v)\end{Bmatrix} \preccurlyeq 0$$

Using implicit relation (A), we get

$$\rho(\mathrm{z},\mathrm{Qv}) \preccurlyeq 0.$$

This gives Qv = z. Therefore Qv = ABu = Pu = STv = z.

Since (P, AB) is occasionally weakly compatible therefore Pu = ABu implies that

PABu = ABPu that is Pz = ABz

Now we show that z is a fixed point of P so we put x=z and y=v in (iii), we get

$$\emptyset\begin{Bmatrix}\rho(\mathrm{Pz},\mathrm{Q}v),\rho(\mathrm{ABz},\mathrm{ST}v),\rho(\mathrm{ABz},\mathrm{Q}v),\\ \rho(\mathrm{ST}v,\mathrm{Pz}),\rho(\mathrm{ABz},\mathrm{Pz}),\rho(\mathrm{ST}z,\mathrm{Q}z)\end{Bmatrix} \preccurlyeq 0$$

$$\emptyset\begin{Bmatrix}\rho(\mathrm{Pz},\mathrm{z}),\rho(\mathrm{z},\mathrm{z}),\rho(\mathrm{z},\mathrm{z}),\\ \rho(\mathrm{z},\mathrm{Pz}),\rho(\mathrm{z},\mathrm{Pz}),\rho(\mathrm{z},\mathrm{z})\end{Bmatrix} \preccurlyeq 0$$

$$\emptyset\begin{Bmatrix}\rho(\mathrm{Pz},\mathrm{z}),0,0,\\ \rho(\mathrm{z},\mathrm{Pz}),\rho(\mathrm{z},\mathrm{Pz}),0\end{Bmatrix} \preccurlyeq 0$$

Using implicit relation (B), we get

$$\rho(\mathrm{z},\mathrm{Pz}) \preccurlyeq 0.$$

This gives Pz = z. Hence Pz = z = ABz.

Similarly (Q, ST) is occasionally weakly compatible we have Qz = STz = z.

Now we show that Bz = z.

Put x = Bz and y = y_n in (iii), we have

$$\emptyset\begin{Bmatrix}\rho(\mathrm{PBz},\mathrm{Q}y_n),\rho(\mathrm{ABBz},\mathrm{ST}y_n),\rho(\mathrm{ABBz},\mathrm{Q}y_n),\\ \rho(\mathrm{ST}y_n,\mathrm{PBz}),\rho(\mathrm{ABBz},\mathrm{PBz}),\rho(\mathrm{ST}y_n,\mathrm{Q}y_n)\end{Bmatrix} \preccurlyeq 0$$

$$\emptyset\begin{Bmatrix}\rho(\mathrm{Bz},\mathrm{z}),\rho(\mathrm{Bz},\mathrm{z}),\rho(\mathrm{Bz},\mathrm{z}),\\ \rho(\mathrm{z},\mathrm{Bz}),\rho(\mathrm{Bz},\mathrm{Bz}),\rho(\mathrm{z},\mathrm{z})\end{Bmatrix} \preccurlyeq 0$$

$$\emptyset\begin{Bmatrix}\rho(\mathrm{Bz},\mathrm{z}),\rho(\mathrm{Bz},\mathrm{z}),\rho(\mathrm{Bz},\mathrm{z}),\\ \rho(\mathrm{z},\mathrm{Bz}),0,0\end{Bmatrix} \preccurlyeq 0$$

Using implicit relation (B), we get

$$\rho(\mathrm{z},\mathrm{Bz}) \preccurlyeq 0.$$

This gives Bz = z.

Since ABz = z therefore Pz = ABz = Bz = Qz = STz = z

Finally we show that Tz = z.

Put x = z and y = Tz in (iii), we get

$$\emptyset\begin{Bmatrix}\rho(\mathrm{Pz},\mathrm{Q}Tz),\rho(\mathrm{ABz},\mathrm{ST}Tz),\rho(\mathrm{ABz},\mathrm{Q}Tz),\\ \rho(\mathrm{ST}Tz,\mathrm{Pz}),\rho(\mathrm{ABz},\mathrm{Pz}),\rho(\mathrm{ST}Tz,\mathrm{Q}Tz)\end{Bmatrix}\preccurlyeq 0$$

$$\emptyset\begin{Bmatrix}\rho(\mathrm{z},Tz),\rho(\mathrm{z},\mathrm{Tz}),\rho(\mathrm{z},Tz),\\ \rho(Tz,\mathrm{z}),\rho(\mathrm{z},\mathrm{z}),\rho(Tz,Tz)\end{Bmatrix}\preccurlyeq 0$$

$$\emptyset\begin{Bmatrix}\rho(\mathrm{z},Tz),\rho(\mathrm{z},\mathrm{Tz}),0,\\ 0,\rho(\mathrm{z},Tz),\rho(\mathrm{z},Tz)\end{Bmatrix}\preccurlyeq 0$$

Using implicit relation (B), we get

$$\rho(\mathrm{z},\mathrm{Tz})\preccurlyeq 0.$$

This gives Tz = z.

Hence ABz = Bz = STz = Tz = Pz = Qz = z. Uniqueness follows easily.

If we put B = T = I, identity map on X, in Theorem 3.1, we have the following:

Corollary 3.1. Let A, S, P and Q is six self mappings of a complex-valued metric space (X,ρ) satisfying the following conditions:

(i) $P(X) \subseteq A(X), Q(X) \subseteq S(X)$,

(ii) The pair (P, A) and (Q, S) share the common (E.A) property.

(iii) For any x, y ∈ X, ∅ in M_6.

$$\emptyset\begin{Bmatrix}\rho(\mathrm{Px},\mathrm{Qy}),\rho(\mathrm{Ax},\mathrm{Sy}),\rho(\mathrm{Ax},\mathrm{Qy}),\\ \rho(\mathrm{Sy},\mathrm{Px}),\rho(\mathrm{Ax},\mathrm{Px}),\rho(\mathrm{Sy},\mathrm{Qy})\end{Bmatrix}\preccurlyeq 0$$

Then the pair (P, A) and (Q, S) have a point of coincidence each. Moreover A, S, P and Q have a unique common fixed point provided both the pairs (P, A) and (Q, S) are occasionally weakly compatible.

As an application of the theorem 3.2., we prove a common fixed point theorem for six finite families of maps on metric space, while proving our results; we utilize definitions of finite families which is natural extension of commutativity condition to two finite families.

Theorem3.2. Let $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_n\}$, $\{S_1, S_2, \ldots, S_p\}$, $\{T_1, T_2, \ldots T_q\}$, $\{P_1, P_2, \ldots P_r\}$,and $\{Q_1, Q_2, \ldots Q_s\}$ be six finite families of self maps of a complex – valued metric space (x, d) such that $A = A_1, A_2, \ldots A_m$, $B = B_1, B_2, \ldots B_n$, $S = S_1, S_2, \ldots S_p$, $T = T_1, T_2, \ldots T_q$, $P = P_1, P_2, \ldots P_r$ and $Q = Q_1, Q_2, \ldots Q_t$ satisfy the following conditions.

(1) P(X) ⊂AB(X)(or Q ⊂ ST(X))

(2) The pair (P, AB) (or (Q, ST)) satisfy property (E.A).

Then the pairs (P, AB) and (Q, ST) have a point of coincidence each .Moreover finite families of self maps P_r ,$A_i B_n$ and Q_t ,S_P ,T_q have a unique common fixed point provided that the pairs of families ({P_r },{A_I },{B_n }and {Q_t},{S_P},{T_q } commute pair-wise for all i= 1,2,…,m, k = 1,2,…n, t = 1,2,…,o, v = 1,2,…,r ,p = 1,2,…, s and q = 1,2,…,x.

Proof. Since self maps A,B,S,T,P and Q satisfy all the conditions of above theorem, the pairs(P,AB) and (Q,ST) have a point of coincidence. Also the pairs of families ($\{P_r\}$,$\{A_i B_n\}$), and ($\{Q_t\}, \{S_p T_q\}$) commute pair wise, we first show that PAB = ABP as

$$\text{PAB} = (P_1P_2 \dots P_r)(A_1A_2 \dots A_m)(B_1B_2 \dots B_n) = (P_1P_2 \dots P_{r-1})(A_1A_2 \dots A_m)(B_1B_2 \dots B_n)$$
$$= (P_1P_2 \dots P_{r-2})(A_1A_2 \dots A_m B_1B_2 \dots B_n P_{r-1}P_r) = \cdots = P_1(A_1A_2 \dots A_m B_1B_2 \dots B_n P_2 \dots P_r)$$
$$= (A_1A_2 \dots A_m)(B_1B_2 \dots B_n)(P_1P_2 \dots P_r) = ABP$$

Similarly one can prove that QST=STQ. Hence, obviously the pair (P, AB) and (Q, ST) are occasionally weakly compatible. We conclude that A, B, S, T, P and Q have a unique common fixed point in X, say z.

Now, one needs to prove that z remains the fixed point of all the component maps.

For this consider

$$\text{A}\,(A_i z) = \big((A_1A_2 \dots A_m)A_i\big)z = \big((A_1A_2 \dots A_{m-1})A_m A_i\big)z$$
$$= (A_1A_2 \dots A_{m-1})(A_i A_m)z = (A_1A_2 \dots A_{m-2})(A_i A_m A_{m-1})z$$
$$= (A_1A_2 \dots A_{m-2})(A_i A_{m-1} A_m)z = \cdots = A_1(A_1A_2 \dots A_m)z$$
$$= (A_1 A_i)(A_2 \dots A_m)z$$
$$=(A_i A_1)(A_2 \dots A_m)z = A_i(A_1A_2 \dots A_m)z = A_i A z = A_i z.$$

Similarly, one can prove that

$$\text{P}\,(B_k z) = B_k(Pz) = B_k z, B(B_k z) = B_k(Bz) = B_k z,$$
$$P(P_v z) = P_v(Pz) = P_v z$$
$$\text{P}\,(A_i z) = A_i(Pz) = A_i z, A(A_i z) = A_i(Az) = A_i z$$
$$P((A_i B_k)z) = (A_i B_k)(Pz) = (A_i(B_k Pz)) = (A_i Pz) = A_i \text{Pz}.$$
$$\text{Q}\,(S_p z) = S_p(Qz) = S_p z, Q\big(T_q z\big) = T_q(Qz) = T_q z,$$
$$Q(Q_t z) = Q_t(Qz) = Q_t z$$
$$Q\big((S_p T_q)z\big) = (S_p T_q)(Qz) = (S_p(T_q Qz)) = (S_p Qz) = S_p z,$$

Which shows that (for all k, i, q, p, v and t) $P_v z$ and $A_i B_k z$ are other fixed point of the pair (P, AB) whereas $Q_t z$ and $S_p T_q z$ are other fixed point of the pair (Q, ST).

As A, B, S, T, P and Q have a unique common fixed point, so, we get

z $= P_v z = A_i z = B_k z = Q_t z = S_p z = T_q z$,

for all v = 1, 2,…,r , i = 1,2,…,m

k=1, 2, …, n , t=1,2,…. , o

p=1, 2, …,s q=1,2,…,x

Which shows that z is a unique common fixed point of$\{P_v\}_{v=1}^{s}$,$\{A_i\}_{i=1}^{m}$, $\{B_k\}_{k=1}^{n}$,$\{Q_t\}_{t=1}^{o}$, $\{S_p\}_{p=1}^{s}$ and $\{T_q\}_{q=1}^{x}$.

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