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STUDY OF BASIC TERMINOLOGY, TECHNIQUES AND PROCESS OF SUMMABILITY

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Abstract

As mathematics is based on principles of reasoning, any slightest deviations from the right track of the flow of mathematical ideas would ultimately end in disharmony. The significance of the notion of summability has been remarkably displayed in various contexts, for example, in analytic continuation, quantum mechanics, probability theory, Fourier Analysis, Approximation Theory and Fixed point theory. The almost summability methods and statistical summability methods has become a dynamic field of research in recent years.In the present article, we gathered some basic definitions and facts which employed in this work. Different notations and terminologies that are used.

Keywords: Terminology, Summability, Process etc.

1. INTRODUCTION

Two substantial tools that contribute to the foundation of current mathematics are, one is the invention of general algorithms of calculus, and the other is the evolution and expansion of infinite series methods. These two streams of expansion, reinforced each other in their simultaneous development, because each served to extend the range of application of the other. The origin of the analysis and derivations of certain infinite series, especially those relating to the arctangent, sine and cosine were not in Europe, but an area in South India that now falls within the state of Kerala [1]. From a area covering about five hundred square kilometers north of Cochin and during the period between the 14th and 16th centuries, there emerged the exploration of infinite series that anticipates similar work of Gregory, Newton and Leibniz over three hundred years. Unlike finite summations, infinite series need tools for mathematical analysis, specifically the notion of limits, to be fully accepted and appreciated. There is valuable documentation that infinite series was hardly used for certain important purposes by Greek Mathematicians like Archimedes, Eudoxous and others in Geometry such as to evaluate the volumes of simple bodies, area bounded by simple curves and other fields [2]. Due to absence of a clear concept of the quantities of the series, Mathematicians considered it as the sum of large number of terms and they accepted that the usual methods relevant to infinite sums would also be relevant to the infinite series. The terms of the series are frequently produced according to a certain rule, such as by a formula, or by an algorithm. As there are an infinite number of terms, this notion is often called an infinite series. In addition to their prevalence in mathematics, infinite series are also broadly used in other quantitative disciplines such as material sciences, software engineering and finance. There are certain infinite series causes paradoxes, defying any satisfactory explanations when arithmetical operations are applied to them without giving rise to any controversy, whereas, it works very well when applied to some others, and made the confusion. Therefore, Mathematicians tried to avoid the use of infinite series as far as possible. But after the discovery of general algorithms of calculus, infinite series becomes a powerful tool and the problem grew more acute as infinite series could no longer be avoided [3]. The only remedial step that was adopted by the mathematicians was to pick out such series on which arithmetic operations where applicable without any doubt and used only those series in the demonstrations. But the sum of an infinite series no longer treated as the arithmetic sum of numbers. It was disappeared soon with the emergence of Cauchy's concept of infinite series. In 1821, A. L. Cauchy, in his book 'Analysis Algebrique' presented a method to determine the sum of infinite series using the limit concept [4].

Let$\sum u_n$ be an infinite series of real or complex terms and $\{s_n\}$ be the sequence of its partial sums, that is,

$$s_n = \sum_{k=0}^{n} u_k.$$

Suppose there exists a definite number s, such that for any $\varepsilon > 0$, there is a number m (depending upon ε) such that,

$$|s_n - s| < \varepsilon, \text{ for all } n > m,$$

then, s is the sum of the series $\sum u_n$, which is also the limit of the sequence $\{s_n\}$. According to Cauchy, only certain specific series for which $\lim_{n\to\infty} s$ exist, are termed as convergent series and have the sums. Moreover, the series that are not convergent called divergent series. Accordingly a divergent series has no sum. Thus, in Cauchy's view, only convergent infinite series, for which sum exists, pertain to the understandable domain of mathematics. The convergency and divergency of infinite series were both in use and no perfection was made till Cauchy [5]. This resulted in many irreconcilable situations and paradoxes to the utter bewilderment of the than mathematicians. Out casting divergent series from the valid domain of the mathematics, Cauchy in one stroke removed all the contradictions and paradoxes involving infinite series. Strengthened further by the fact that the Cauchy's concept of a sum, its acceptability was immediate and profound. It began to be universally held that the problem of the sum of an infinite series had fully and finally been resolved [6].

In course of time, it led mathematicians to wind up that the Cauchy's method of assigning sum to an infinite series was a far reaching impart and quite effective as far as it went, but indeed that divergent series were not that satanic as they were earlier made out to be. All these stirred the imagination of several curious mathematicians to examine deep into the character of the sum of an infinite series, over and above that of the Cauchy for assigning sum. Persistent efforts made by a number of famous mathematicians led to the development of alternative methods, closely related to that of Cauchy to associate sum to non-convergent series too. Especially to the series whose partial sums oscillate. In fact, during the early decades of the nineteenth century, several additional methods of assigning sum to infinite series, consistent with that of Cauchy were invented by mathematicians. These methods of summation were termed as Summability Methods [7].

Some of the most important methods of summability are those which are associated with the names of the great mathematicians. By the third decade of the twentieth century, a very rich and useful theory of summability had emerged. This theory not only set right many disharmonies noticed in the field of infinite series, but also influenced and enriched other allied fields too. In fact, this theory found more applications, even in such isolated fields as the "Probability theory" and "Theory of Numbers" [8].

1.2 Basic Terminologies

We present here a comprehensive scheme of some ideas and terminologies in summability theory, which have relevance to our study.

Definition 1.2.1. A summability method or summation method is a transformation from the set of sequence of partial sums of a series to a value.

In other words, in the extended sense, summability theory or in short summability is the theory of assignment of limits to infinite series, which is elementary in Analysis, Theory of Function Theory, Functional Analysis and Topology [9].

Definition 1.2.2. A series$\sum_{n=0}^{\infty} u_n$ is said to be summable by the method P (P-summable), if it assigns a sum to the infinite series by the method P (P-transform). Also, here we say $\sum_{n=0}^{\infty} u_n \in$ P. Similarly, if $\sum_{n=0}^{\infty} u_n \notin$ P, then $\sum_{n=0}^{\infty} u_n$ is not summable by the method P

Alternatively, we say that, the series $\sum_{n=0}^{\infty} u_n$ is P-convergent to l, if the sequence of partial sum $\{s_n\}$ is P-convergent to l.

Furthermore, if the new sum of a series coincides with Cauchy's sum, the method will be more useful. Accordingly, we define the followings [10];

- A summability method P is called conservative, if the convergence of a series implies its summability by the method P.
- A summability method P is called absolutely conservative, if absolute convergence of a series implies its absolute summability by the method P.
- A summability method P is called regular, if it is conservative and preserves the sum of the convergent series (it coincides with Cauchy's sum). In other words, if it sums all convergent series to its Cauchy's sum, then it is called regular.
- A summability method P is said to be absolutely regular, if it is absolutely conservative and regular.
- A summability method is called consistent, if it assigns same sum to the same series.
- Two summability methods are consistent, if they would not sum a series to two different sums.

2. BASIC TECHNIQUES

The basic technique behind all the summability methods is to transform a given infinite series, or sequence of its partial sum into another series or sequence on which Cauchy's method is also applicable. The transformation chosen is generally linear and is such that it preserves the Cauchy's sum when applied to convergent series. Further, the transformation to be worthy, it should be such as to transform some divergent series into one on which Cauchy's method of assigning sum can be applied [11]. Thus, if T is a transformation which corresponds a summability method, then it should satisfy the properties as follows:

(i) If $\sum u_n$is a convergent series with sum s, then (T Pu_n) should also be convergent and has the same sum s,

(ii) If $\sum u_n$and $\sum v$are two series and λ and μ are any two real (or complex) constants, then $T(\lambda \sum u_n + \mu \sum v_n) = \lambda(T \sum u_n) + \mu(T \sum v_n)$, and

(iii) The T-method can find sum to at least one infinite series for which Cauchy's method fails.

Here, the conditions (i), (ii) and (iii) are called regularity conditions, linearity conditions and range condition respectively.

3. BASIC PROCESS

All the summability methods can be classified broadly into two basic general processes, usually known as T-Process (Methods based upon sequence-to-sequence transformations) and φ-Process (Methods based upon sequence-to-function transformations) [12].

3.1 T-Process

T-Process is a summability method, where the sequence of partial sums of infinite series or sequences is transformed into another sequence.

Let $\sum u_n$ be an infinite series with a sequence of partial sums $\{s_n\}$ and this series is said to have the Cauchy's sum, if lim sn = s (finite number). Let T be a transformation (linear), and let $\{t_n\} = \{T(s_n)\}$. Then, the T-method consists in the development of an supplementary sequence $\{t_n\}$, obtained by the sequence-to-sequence transformation. The series $\sum u_n$ is summable by T-method to the sum s, iff $\lim_{n\to\infty} t_n = s$.

Moreover, we say that the series Pu_n is absolutely convergent, if $\sum |u_n| < \infty$, which is similar as:

$$\sum |s_n - s_{n-1}| < \infty.$$

that is to say, the sequence $\{s_n\}$ is of bounded variation, (written here after as B.V.). Following the same analogy, $\sum u_n$ is said to be absolutely summable by T-method or simply |T|- summable, iff the auxiliary sequence $\{t_n\}$ is of B.V., that is,

$$\sum |t_n - t_{n-1}| < \infty.$$

T-method is known to be absolutely regular, iff

$$\{s_n\} \in B.V. \Rightarrow \{t_n\} \in B.V.$$

Characterization of T-Process

It would be obvious from the above discussion that in any plan of finding the sum of an infinite series by a T-Process, it is the sequence of partial sums of an infinite series that performs a vital role. Likewise, since each series has a specific representation in terms of its partial sums and vice-versa, and further, that the series is summable if its sequence of partial sums is limitable. In the theory of summability, it is as good to work on the sequence of partial sums as on the series itself. Also, it is suitable to impact transformation on a sequence than in a series. Thus, T-Process is more convenient to operate on the sequence instead of the series.

Regularity Condition

The necessary and sufficient conditions for regularity of T -summability method, expressed by a triangular matrix (a_n,k) was introduced by a well-known mathematician. Hence, T-method is represented by,

$$t_n = \sum_{k=0}^{n} a_{nk} s_k, \quad \text{for } n = 0, 1, 2, \cdots$$

is regular iff,

(i) $\sup \sum_{k=0}^{\infty} |a_{n,k}| < \infty$,

(ii) $\lim_{n\to\infty} a_{n,k} = 0$

(iii) $\lim_{n\to\infty} (\sum_k a_{n,k}) = 1$.

Range Condition

Let T be a summability method and F(T) denotes the set of all sequences that are summable by T. Also, F(T) is known as the field of convergence of T, and is given by

$$F(T) = \{(s_n) : T\{(s_n)\} \in c\}.$$

Hence, T satisfies the range condition iff $c \subseteq F(T)$.

Conservative Process

If the convergence of $\{s_n\}$ implies the convergence of $\{t_n\}$, then T -method is called conservative.

3.2 φ-Process

φ-process is a summability method of sequence-to-function transformation type, where the sum of an infinite series is transformed into a continuous function.

Let $\sum u_n$ be an infinite series with its sequence of partial sums denoted by $\{s_n\}$ and let $\{\varphi_n(x)\}$ be a sequence of continuous functions with variable x, and is such that $\sum \emptyset_n(x) s_n$ exists in a suitable interval of x.

Thus, if t(x) is a function transform obtained by the φ-method, then it is given by

$$t(x) = \sum_n \phi_n(x) s_n$$

where x is a continuous parameter. Then, the corresponding integral transformation is given by

$$t(x) = \int_0^{\infty} \phi(x, \zeta) s(\zeta) d\zeta.$$

where$\varphi_n(x, \zeta)$ or $[\varphi(x, \zeta)]$ is defined over a suitable interval of x (or of x and ζ) and s(ζ) = sn for n = ζ.

Also, as in the case of T-method, $\sum u_n$ is summable to s by the method φ, iff

$$\lim_{x\to\infty} t(x) = s.$$

where a is the boundary point of the domain of x:

Likewise, the series $\sum u_n$ is absolutely summable by the method φ, that is |φ|- summable, if t(x) ∈ B.V., in a given domain of x. Also, the φ-method is regular, if

$$\lim_{n\to\infty} s_n = s \Rightarrow \lim_{x\to\infty} t(x) = s.$$

The necessary and sufficient conditions for regularity of this φ -method are analogous to those of the T-method are:

(i) $\sum \emptyset_n(x)$ is convergent (for every x ≥ 0),

(ii) $\sum \emptyset_n(x) < k$, (K is independent of x ≥ 0),

(iii) $\lim_{n\to\infty} \emptyset_n(x) = 0$, for every x,

(iv) $\lim_{x\to a} \emptyset_n(x) = 1$(where a is the boundary point in the domain of x).

4. CONCLUSION

So it is concluded that the usual methods of summability may be viewed as the generalization of Cauchy's concept of convergence. Further, the concept of ordinary convergence has been extended into that of summability, commonly termed as ordinary summability, the idea of absolute convergence too has been enhanced into the concept termed as absolute summability.Though summability methods are devised to associate a sum in a logical way to some non-convergent series; however it will not worthwhile, if it fails to assign a sum of series which is convergent in Cauchy's sense.

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