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Discrete Semi Group of Initial Boundary Value Problems

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Abstract

The discrete semi group of initial-BVP for first order PDEs is obtainable in the present study. We derive an adapted problem posed on a bounded domain whose explanation is identical to the solution of the novel problem on a smaller bounded domain for the IVP.On the smaller bounded domain, numerical solution to the adapted problem converges to the solution to the original issue. We offer discrete semi group approximations for the IVP by decomposing it into two problems, each of which generates a semi group.

Keywords: SemiGroup, Initial Value Problem

Introduction

In mathematics, in the field of partial differential equations, an initial value problem is a partial differential equation together with a specified value called the initial condition of the unknown function at a given point in the domain of the solution. In physics and other fields, addressing an initial value issue is a common part of modelling a system.

Boundary value issues are often used to express difficulties involving the wave equation, such as the identification of the normal nodes. Problems involving boundary values are quite similar to those involving starting values. There are no conditions provided at the extremes of the independent variable in a boundary value problem, whereas all conditions are specified in an initial value problem at the same value of the variable in the equation. Both an initial value and a boundary value issue must be well-posed before they can be used in practical applications. First-order hyperbolic partial differential equations are well-documented. The numerical approximation approaches for initial value and initial boundary value issues have seen a great deal of development.

The stability of finite difference schemes for first order hyperbolic initial-boundary value problems with vector values functions in L2(IR+, IRN) was examined by Gottlieb et al (1987), Bo (1998), and Coulombel (2009). Discrete approximations to the initial-boundary value issue were investigated in 1988 by Warming and Beam.

$$U_t = aU_x, 0 \leq x \leq A, t \geq 0,$$

U(x,0) = u(x), 0$\leq x \leq A$,

U(A,t) = v(t), $\geq 0,$ (1)

The stability of finite difference schemes for first order hyperbolic initial-boundary value problems with vector values functions in L^2(IR+, IRN) was examined by Gottlieb et al (1987). Discrete approximations to the initial-boundary value issue were investigated in 1988 by Warming and Beam.

$$u_t + au_x = 0, x \in R, t \in R^+$$

u(x,0) = $u_0(0)$, x$\in R$ (2)

For limited discontinuous starting functions u_0, for the development of numerical schemes for the beginning and boundary value issue, these research were motivated in which the initial

$$u_t = a(x)u_x(x) = 0, x \in R^+, t \in R^+$$

u(x,0) = $u(x)$, x$\in R^+$ (3)

condition is defined as some given function with the initial condition being defined as a(x) > 0 for all values of x$\in R^+$. When waves travel in a homogeneous medium, Equation (3) serves as the model.

The Initial-Boundary Value Issue (IBVP) is the name given to the second model problem.

$$U_t = -aU_x, x \in [0,1], t \in R^+$$

$$U(x,0) = u(x), x \in [0,1],$$

$U(0,t) = v(t), t \in R^+$ (4)

Assume that a > 0 and that a boundary condition v(t) is provided when x = 0 in this scenario. Information travels from left to right, thus u∈C[0, 1] and v∈C[0, C], which meet the compatibility criterion of u(0) = v. (0).

When solving IVP (3) and IBVP (4), semigroup theory was employed extensively. The initial-boundary value issues may now be solved in an elegant way thanks to semi group theory.

Theorem. Let X be a Banach space with a norm of|| || . X For example, assume that D(A) is dense in X, a linear map, A: D(A) →X is the range of A: A: A: A: A: A: A: A: Think of the Banach spaces X_n as being Banach spaces with norms that are less than or equal to one. In addition, there are bounded linear operators that are P_n: XX_n and E_n:X_n: X_n

i. $||P_n|| \leq C_1$, $||E_n|| \leq C_2$, with C_1 and C_2 are constants independent of n.
ii. $||P_n x||_n \rightarrow ||x||\ as\ n \rightarrow \infty\ for\ every\ x \in X.$
iii. $||E_n P_n x - x|| \rightarrow 0\ as\ n \rightarrow \infty\ for\ every\ x \in X.$
iv. $P_n E_n = I_n, where\ I_n\ is\ the\ identity\ operator\ on\ X_n.$

Let F (τn) be a sequence of bounded linear operators from Xn into Xn satisfying

$||F(\tau_n)^k|| \leq 1$ (5)

Besides, the bounded linear maps

$\lim_{n\rightarrow\infty} E_n A_n P_n x = Ax$ (6)

Moreover, if knτn → t as n → ∞, then

$\lim_{n\rightarrow\infty} ||F(\tau_n)^{k_n} P_n x - P_n S(t)x||_n = 0$ (7)

In the sequel, the term solution refers to a generalized solution in an appropriate sense

For$\alpha=(\alpha_0,\alpha_1,...,\alpha_k)$,*thenotation*$\alpha(i)=\alpha_i$*isused.*

For x∈ R, $[x] = \sup\{n \in Z: n \leq x\}$

Exact Solution for the Initial Value Problem

It is well known that the solution to (3.3) is given by

u(x, t) = u($\beta^{-1}(t + \beta(x)$)

where$\beta(x) = \int_0^x \frac{d\xi}{a(\xi)}$

On a bounded domain, the goal was to numerically solve (3) using the non-bounded solution u(x, t) of (3), which was not necessarily constrained. This conclusion is made possible by the following theorem.

Theorem. Assume that a ∈C[0, ∞) and a(x) > 0 for all x ∈ IR^+. Let M > 0 and T > 0. Define a_M : [0, M] → IR^+ as

$a_M(x)$ =a(x), 0$\leq x \leq M - \frac{1}{M}$

=a(M-1/M)$\sqrt{M(M-x)}, M - 1/M \leq x \leq M$

and let f ∈ C[0, M]. The solution to the problem

$$\frac{\partial V}{\partial t} = a_M(x)\frac{\partial V}{\partial x}, 0 \leq t \leq T, 0 \leq x \leq M$$

V(x,0) = f(x), 0$\leq x \leq M,$

V(M,t) = f(M) (8)

exists, unique and is given by

V(x,t) = f($\beta_M^{-1}[Min\big(t + \beta_M(x), \beta_M(M)\big)$], where

$$\beta_M(x) = \int_0^x \frac{d\xi}{a(\xi)}, 0 \leq x \leq M - \frac{1}{M},$$

$$= \int_0^{M-1/M} \frac{d\xi}{a\xi} + \int_{M-1/M}^{x} \frac{d\xi}{\alpha(M-\frac{1}{M})\sqrt{M(M-x)}}, M - \frac{1}{M} \leq x \leq M$$

Further,

$S_t f(x)$ =f($\beta_M^{-1}[Min\big(t + \beta_M(x), \beta_M(M)\big)$

defines a contraction semigroup on C[0, M] whose generator is given by

D(A) = [g∈ $C[0, M]$: g' ∈ $C[0, M)and$ $\lim_{x\to M} a_M(x)g'x) = 0$}

And

Ag(x) =$a_M(x)g'x$

Ag(M) = 0

Further, choosing M > N such that

$$sup_{t\in[0,T],x\in[0,N]}\big(t + \beta(x)\big) < b\left(M - \frac{1}{M}\right),$$

V(x,t) = u(x,t), (x,t)∈ $[0, N] * [0, T]$ (9)

provided f ∈ C[0, M] is the restriction of u to [0, M].

Proof: Define for $t \geq 0$, $T_t : [0, M] \to [0, M]$ as

$S_t f(x) = \beta_M^{-1}[Min\big(t + \beta_M(x), \beta_M(M)\big)$

$T_s * T_t x = \beta_M^{-1}[Min\big(s + \beta_M(T_t x), \beta_M(M)\big)$

$= \beta_M^{-1}[Min(s + \beta_M(B_M^{-1}[Min(t + \beta_M(x)), \beta_M(M)]), \beta_M(M))]$

$=\beta_M^{-1}[Min\big(s + t + \beta_M(x), \beta_M(M)\big)]$

$=_{Ts+t}$x

Also, it is easy to say that St is a semigroup, since $S_t f(x) = f(T_t x)$.

It is obvious that $kS_t fk \leq kfk$ and hence S_t is a contraction semigroup.

Now, by Hille-Yosida Theorem, if B is the generator of S_t then

$$(I - B)^{-1}h(x) = \int_0^\infty e^{-t} S_t h(x) dt$$

$$= \int_x^M e^{\beta_M(x) - \beta_M(y)} \frac{h(y)}{a_M(y)} dy + h(N) e^{\beta_M(x) - \beta_M(y)}$$

Where $y = \beta_M^{-1}(t + \beta_M(x))$

Now, consider the differential equation

$f(x) - a_M(x) f'(x) = h(x), x \in [0, M),$

f(M) = h(M)

which is equivalent to

$f(x) - a(x) f'(x) = h(x), x \in [0, M],$

$$\lim_{x \to M} a(x) f'(x) = 0$$

for every h ∈ X, there is a unique solution f ∈ D(A) to the above differential equation which is given by

$f(x) = \int_x^M e^{\beta_M(x) - \beta_M(y)} \frac{h(y)}{a_M(y)} dy + h(N) e^{\beta_M(x) - \beta_M(y)}$

Hence it can be shown for the operators A and B, $(I - A)^{-1} = (I - B)^{-1}$. From this, one can easily conclude that D(A) = D(B) and for all g ∈ D(A), $B_g = A_g$.

As β is a strictly increasing function by (9), for t∈ [0, T] and x ∈ [0, N] then

$x \leq \beta^{-1}\big(t + \beta(x)\big) < M - 1/M$

Hence

$$\beta \beta^{-1}\big(t + \beta(x)\big) = \beta_M \beta_M^{-1}(t + \beta_M(x))$$

From this, it is concluded that $S_t f(x) = V(x, t) = u(x, t)$ for all x ∈ [0, N] and t ∈ [0, T].

Convergent Numerical Scheme for the Initial Value Problem and Initial-boundary Value Problem

First and boundary value convergent numerical schemes are explained in this section. It is possible to solve the initial value issue by posing it on a smaller bounding box, and then solving it on a larger bounding box with the same answer. The numerical solution to the modified problem converges to the solution of the original issue in the smaller constrained region. The discrete semigroup approximations for the initial-boundary value issue may be presented by splitting it into two separate problems, each of which yields a semigroup.

A Convergent Numerical Scheme for the IVP

Using the initial value problem (3), one may get M > N and an initial value problem posed on [0, M] [0, T] whose solution precisely matches the solution of (3) on [0, T]. On [0, M] [0, T], one builds a finite difference scheme that converges to the solution of the issue given in (3.3) on [0, N] [0, T].

This is made possible by the following theorem.

Theorem: Let, X is C[0, M] and A is the same as in Assume $X_n = R^{n+1}$, where n is the number of items in X_n. The supremum norm is used to standardise the spaces X and X_n. We'll get to it in a moment,

$P_n : X \to X_n$ as $(P_n f)_i = f(iM/n)$, i = 0, 1, . . . , n.

$E_n : X_n \to X$ as

$E_n(\alpha)$ is the piecewise linear function with $En(\alpha)(iM/n) = \alpha_i$.

Let

$$\tau_n = \frac{1}{2nsup_{x\in[0,M]}|a(x)|}$$

Define an operator $F(\tau_n) : X_n \to X_n$ as

$$(F(\tau_n)\alpha)i = \left(1 - n\tau_n a_M\left(\frac{iM}{n}\right)\right)\alpha_i + n\tau_n a_M\left(\frac{iM}{n}\right)\alpha_{i+1}, i = 0,1,\dots,n-1$$

$$= \alpha_n, i = n$$

Choosing $kn = t/t_n$, it can be shown that

$$\|F((\tau_n)^{k_n} P_n f - P_n S(t) f\|_n \to 0 \; as \; n \to \infty$$

In particular, fixing t ∈ [0, T] and x ∈ [0, N],

$$\lim_{n\to\infty} F(\tau_n)^{k_n} P_n f\left(\left[\frac{nx}{M}\right]\right) = u(x,t)$$

where u(x, t) is the solution to (3.3).

A Convergent Numerical Scheme for the IBVP

The theory of semigroups cannot be directly applied to an initial-boundary value issue. There are discrete semigroups that can approach this semigroup, however it can be broken down into two difficulties.

This conclusion is made possible by the following theorem.

Theorem. Let X and Y be as in above Theorem. Take $X_n = R^n$ and $Y_n = R^{n+1}$. Define the following quantities

$$\tau_n = \frac{1}{n[2a+1],}$$

$$k_n = [nt[2a+1],$$

b = 1/a

$$\eta_n = \frac{1}{n[2b+1],}$$

And

$$\xi_n = [nx[2b+1],$$

Further, define $P_n : X \to X_n$ as $[P_n f]_i = f(i/n)$, i = 1, . . . , n and $E_n : X_n \to X$ as $E_n(\alpha)$ being the piece-wise linear function with $(E_n(\alpha))(0) = 0$ and $(E_n(\alpha)(i/n) = \alpha_i$, i = 1, 2, . . . , n and the operator

$F(\tau_n) : X_n \to X_n$

As

$(F(\tau_n)\alpha i = \left(1 - \frac{a}{[2a+1]}\right)\alpha_i + \frac{a}{[2a+1]}\alpha_{i-1}, i = 2,3,\ldots,4$

$= \left(1 - \frac{a}{[2a+1]}\right)\alpha_i, i = 1$

Besides, take $Q_n : Y \to Y_n$ as

$[Q_n f]_l = f(lT/n)$, l = 0, 1, . . . , n

and$H_n : Y_n \to Y$ as

$H_n(\alpha)$ being the piece-wise linear function with $(H_n(\alpha)(lT/n) = \alpha_l$, l = 0, 1, 2, . . . , n.

Define an operator

$G(\eta_n): Y_n \to Y_n$

$(G(\eta_n)\alpha)l = \left(1 - \frac{a}{[2b+1]}\right)\alpha_l + \frac{a}{[2b+1]}\alpha_{l-1}, l = 2,3, \dots ,4$

$= \alpha_0, l = 0$

Then for the initial value problem (4),

$$\log_{n\to\infty}(F(\tau_n)^{k_n})P_n u_0)(\lfloor nx \rfloor) + \left(G(\eta_n)^{\xi_n} Q_n v\right)\left(\left\lfloor \frac{nt}{T} \right\rfloor\right) = U(x,t)$$

for fixed x and t.

Conclusion:

The start and initial-boundary value issue for first-order PDEs in unbounded domains was addressed in this paper. Exact solutions for beginning and initial-boundary value problems were sought in the first half of this study, while the second part of this study was focused on the convergence of numerical schemes for IVPs and IBVPs.Finally, the research team presented the following methods for solving infinite-delay differential equations. An infinite-delay neutral delay differential equation has been solved numerically and its asymptotic stability has been explored in the first phase. PDEs with infinite delay were semi-discretized and discrete semigroup approximation for first order PDEs in unbounded domains were produced in the second phase of the study programmer.

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