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Application of (4,3)- jection operator in exponential function

Anand kumar
Scholar, Department of Mathematics, J. P. University, Chapra
841301, Bihar, India.

Abstract:- In this paper, we use (4,3)- jection operator which is the generalisation of projection operator, projection defined in G.F. simmons [1]. We have define e^B, where B is a (4,3)- jection operator on a vector space V. We calculate the value of e^{nB}, n being any integer. We consider some theorems. We define hyperbolic function involving a (4,3)-jection B .We deduce properties of hyperbloic function such as $\cos h2B = \cos h^2 B + \sin h^2 B$. We next define sinB, cosB etc and establish formulae analogous to results in trigonometry. We introduce differentiation of e^{xB} and show differentiation of $\cos(hxB)$ and $\sin(hxB)$.

Key words:- vector space, projection operator, exponential function, (4,3)-jection operator.

1. Introduction :- Let B be a linear operator on a vector space V. Then B is (4,3)-jection if $B^4 = B^3$. In this paper we defined exponential function of (4,3)- jection operator.

2. Definition and some result:- Let B be a (4,3)- jection operator on a vector space V. [2].

i.e. $B^4 = B^3$

We define,

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \frac{B^4}{4!} + \frac{B^5}{5!} + \frac{B^6}{6!} + \text{--------}$$

Now , $e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \frac{B^3}{4!} + \frac{B^3}{5!} + \frac{B^3}{6!} +$ -------- (As $B^4 = B^3$ from above)

$= I + B + \frac{B^2}{2!} + B^3[\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \text{--------}]$

$= [I + B + \frac{B^2}{2!} + B^3 \{(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \text{--------}) - (1 + 1 + \frac{1}{2!})\}]$

$e^B = I + B + \frac{B^2}{2!} + B^3\ [e - \frac{5}{2}]$--------------------------------------(2.1)

Therefore, $(e^B)^2 = [I + B + \frac{B^2}{2!} + B^3\ \{e - \frac{5}{2}\}]^2$

$= [I^2 + B^2 + \left\{\frac{B^2}{2}\right\}^2 + \left\{B^3\ (e - \frac{5}{2})\right\}^2 + 2.I.B + 2.B.\frac{B^2}{2} + 2.\ \frac{B^2}{2}.B^3\ (e - \frac{5}{2}) + 2.\ B^3\left(e - \frac{5}{2}\right).I + 2.I\ .\frac{B^2}{2} + 2.B.B^3\ (e - \frac{5}{2})]$

$= [I^2 + B^2 + \frac{B^4}{4} + B^6(e - \frac{5}{2})^2 + 2.B + B^3 + B^5\ (e - \frac{5}{2}) + 2.\ B^3\left(e - \frac{5}{2}\right) + B^2 + 2B^4\ (e - \frac{5}{2})]$

$= [I + B^2 + \frac{B^3}{4} + B^3(e - \frac{5}{2})^2 + 2.B + B^3 + B^3\ (e - \frac{5}{2}) + 2.\ B^3\left(e - \frac{5}{2}\right) + B^2 + 2B^3\ (e - \frac{5}{2})]$

$= [I + 2.B + 2B^2 + B^3\{\frac{1}{4} + e^2 + \frac{25}{4} - 2.e.\frac{5}{2} + 1 + e - \frac{5}{2} + 2.e - 5 + 2.e - 5\}]$

$\therefore\ (e^B)^2 = [I + 2.B + 2B^2 + B^3\{e^2 - 5\}]$

Now Also, $(e^B)^3 = e^B.(e^B)^2$

$= [I + B + \frac{B^2}{2!} + B^3\ \{e - \frac{5}{2}\}].[I + 2.B + 2B^2 + B^3\ \{e^2 - 5\}]$

$= I + 2.B + 2B^2 + B^3(e^2 - 5) + B + 2B^2 + 2B^3 + B^4(e^2 - 5) + \frac{B^2}{2} + \frac{2B^3}{2} + \frac{2B^4}{2} + \frac{B^5}{2}(e^2 - 5) + B^3(e - \frac{5}{2}) + 2.B^4(e - \frac{5}{2}) + 2B^5(e - \frac{5}{2}) + B^6(e - \frac{5}{2}).(e^2 - 5)$

$= I + 2.B + 2B^2 + B^3(e^2 - 5) + B + 2B^2 + 2B^3 + B^3(e^2 - 5) + \frac{B^2}{2} + B^3 + B^3 + \frac{B^3}{2}(e^2 - 5) + B^3(e - \frac{5}{2}) + 2.B^3(e - \frac{5}{2}) + 2B^3(e - \frac{5}{2}) + B^3(e^3 - 5e - \frac{5}{2}e^2 + \frac{25}{2})$

$\therefore\ (e^B)^3 = [I + 3.B + \frac{9}{2}B^2 + B^3(e^3 - \frac{17}{2})]$

Therefore $(e^B)^4 = e^B.(e^B)^3$

$= [I + B + \frac{B^2}{2!} + B^3\ \{e - \frac{5}{2}\}].[I + 3.B + \frac{9}{2}B^2 + B^3(e^3 - \frac{17}{2})]$

$= [I + B + \frac{B^2}{2!} + B^3\{e - \frac{5}{2}\} + 3B + 3B^2 + \frac{3}{2}B^3 + 3B^4\{e - \frac{5}{2}\} + \frac{9}{2}B^2 + \frac{9}{2}B^3 + \frac{9}{2}B^4 + \frac{9}{2}B^5\{e - \frac{5}{2}\}$

$+ B^3\{e^3 - \frac{17}{2}\} + B^4\{e^3 - \frac{17}{2}\} + \frac{B^5}{2}\{e^3 - \frac{17}{2}\} + B^6\{e^3 - \frac{17}{2}\}\{e - \frac{5}{2}\}]$

$= [I + 4B + 8B^2 + B^3\{e^4 - 13)]$

$\therefore\ (e^B)^4 = [I + 4B + 8B^2 + B^3\{e^4 - 13)]$

Thus $(e^B)^4 \neq (e^B)^3$

Therefore e^B is not a (4,3)−jection.

Next, we assume

$(e^B)^n = [I + nB + \frac{n^2}{2}B^2 + B^3\{e^n - \frac{n(n+2)}{2} - 1)]$

$e^{nB} = [I + nB + \frac{n^2}{2}B^2 + B^3\{e^n - \frac{n^2+2n+2}{2})]$---(2.2)

Now, $(e^B)^{n+1} = (e^B)^n \times (e^B)$

Then, $[I + B + \frac{B^2}{2!} + B^3\ \{e - \frac{5}{2}\}]^{n+1}$

$= [I + B + \frac{B^2}{2!} + B^3\ \{e - \frac{5}{2}\}]^n \times [I + B + \frac{B^2}{2!} + B^3\ \{e - \frac{5}{2}\}]$

$= [I + nB + \frac{n^2}{2}B^2 + B^3\{e^n - \frac{n(n+2)}{2} - 1)]\ [I + B + \frac{B^2}{2!} + B^3\ \{e - \frac{5}{2}\}]$

$= [I + nB + \frac{n^2}{2}B^2 + B^3(e^n - \frac{n(n+2)}{2} - 1) + B + nB^2 + \frac{n^2}{2}B^3$

$+ B^4(e^n - \frac{n(n+2)}{2} - 1) + \frac{B^2}{2} + n\frac{B^3}{2} + \frac{n^2}{2}B^4 + \frac{B^5}{2}(e^n - \frac{n(n+2)}{2} - 1)$

$+ B^3\ (e - \frac{5}{2}) + nB^4\ (e - \frac{5}{2}) + \frac{n^2}{2}B^5\ (e - \frac{5}{2}) + B^6\ (e - \frac{5}{2})(e^n - \frac{n(n+2)}{2} - 1)]$

$= I + (n+1)B + \frac{B^2(n+1)^2}{2} + B^3(e^{n+1} + e^n\left(2 + \frac{1}{2} + \frac{5}{2}\right) + e(n + \frac{n^2}{2} - \frac{n(n+2)}{2}) + (-\frac{n(n+2)}{2} + \frac{n^2}{2} - \frac{n(n+2)}{2} + \frac{n}{2} + \frac{n^2}{4} - \frac{n(n+2)}{4} - \frac{5n}{2} - \frac{5n^2}{4} + \frac{5n(n+2)}{4}) - \frac{5}{2}]$

$= I + (n+1)B + \frac{B^2(n+1)^2}{2} + B^3(e^{n+1} + \frac{-2n^2-8n}{4} - \frac{5}{2})$

$= I + (n+1)B + \frac{B^2(n+1)^2}{2} + B^3(e^{n+1} - \frac{n^2+4n+5}{2})$

$= I + (n+1)B + \frac{B^2(n+1)^2}{2} + B^3(e^{n+1} - \frac{n^2+4n+3}{2} - 1)$

$= I + (n+1)B + \frac{B^2(n+1)^2}{2} + B^3(e^{n+1} - \frac{(n+1)(n+3)}{2} - 1)$

Thus $(e^B)^{n+1} = I + (n+1)B + \frac{B^2(n+1)^2}{2} + B^3(e^{n+1} - \frac{(n+1)(n+3)}{2} - 1)$

By Induction, for any positive integer n.

Now, we discuss some particular cases

If $(e^B)^n = [I + B + \frac{B^2}{2!} + B^3\ \{e - \frac{5}{2}\}]^n$; (from 2.1)

$e^{nB} = [I + nB + \frac{n^2}{2}B^2 + B^3\{e^n - \frac{n^2+2n+2}{2})]$ (from 2.2)

If we putting n = 0 in (2.2) then,

$e^{0B} = [I + 0.B + \frac{0^2}{2}B^2 + B^3\{e^0 - \frac{0^2+2.0+2}{2})]$

$e^0 = I$ where 0 is zero operator

So relation holds when n = 0

We take n = −n in (2.2) then,

i.e. when n is negative integer.

$e^{-nB} = [I + (-n)B + \frac{(-n)^2}{2}B^2 + B^3\{e^{-n} - \frac{(-n)^2+2(-n)+2}{2})]$

$e^{-nB} = [I - nB + \frac{n^2}{2}B^2 + B^3\{e^{-n} - \frac{n^2-2n+2}{2})]$---------------------------------(2.3)

We take B = I in (2.2) then,

$e^{nI} = [I + nI + \frac{n^2}{2}I^2 + I^3\{e^n - \frac{n^2+2n+2}{2})]$

$= I[1 + n + \frac{n^2}{2} + \{e^n - \frac{n^2+2n+2}{2})]$
$e^{nI} = Ie^n$ or $e^n I$

Theorem(2.I):-

$$e^{nB} \times e^{-nB} = I$$

Proof: we have,

$$e^B = I + B + \frac{B^2}{2!} + B^3\ [e - \frac{5}{2}]$$

$e^{nB} = [I + nB + \frac{n^2}{2}B^2 + B^3\{e^n - \frac{n^2+2n+2}{2})]$

Now, $e^{nB} \times e^{-nB}$

$= [I + nB + \frac{n^2}{2}B^2 + B^3(e^n - \frac{n^2+2n+2}{2})]\ [I - nB + \frac{n^2}{2}B^2 + B^3(e^{-n} - \frac{n^2-2n+2}{2})]$

$= [I - nB + \frac{n^2}{2}B^2 + B^3(e^{-n} - \frac{n^2-2n+2}{2}) + nB - n^2B^2 + \frac{n^3}{2}B^3 + nB^4(e^{-n} - \frac{n^2-2n+2}{2}) + \frac{n^2}{2}B^2 - \frac{n^3}{2}B^3 + \frac{n^4}{4}B^4 + \frac{n^3}{2}B^5(e^{-n} - \frac{n^2-2n+2}{2}) + B^3(e^n - \frac{n^2+2n+2}{2}) - nB^4(e^n - \frac{n^2+2n+2}{2}) + \frac{n^2}{2}B^5(e^n - \frac{n^2+2n+2}{2}) + B^6(e^n - \frac{n^2+2n+2}{2})\ (e^{-n} - \frac{n^2-2n+2}{2})$

$= [I + \frac{n^2}{2}B^2 + B^3(e^{-n} - \frac{n^2-2n+2}{2}) - n^2B^2 + \frac{n^3}{2}B^3 + nB^3(e^{-n} - \frac{n^2-2n+2}{2}) + \frac{n^2}{2}B^2 - \frac{n^3}{2}B^3 + \frac{n^4}{4}B^4 + \frac{n^3}{2}B^3(e^{-n} - \frac{n^2-2n+2}{2}) + B^3(e^n - \frac{n^2+2n+2}{2}) - nB^3(e^n - \frac{n^2+2n+2}{2}) + \frac{n^2}{2}B^3(e^n - \frac{n^2+2n+2}{2}) + B^3\{e^n e^{-n} - e^n\left(\frac{n^2-2n+2}{2}\right) - e^{-n}\left(\frac{n^2+2n+2}{2}\right) + \frac{1}{4}(n^2 + 2n + 2)(n^2 - 2n + 2)\}]$ (As $B^4 = B^3$)

$= [I + B^2(\frac{n^2}{2} - n^2 + \frac{n^2}{2}) + B^3\{e^{-n} - \frac{n^2-2n+2}{2} + \frac{n^3}{2} + n(e^{-n} - \frac{n^2-2n+2}{2}) - \frac{n^3}{2} + \frac{n^4}{4} + \frac{n^3}{2}(e^{-n} - \frac{n^2-2n+2}{2}) + (e^n - \frac{n^2+2n+2}{2}) - n(e^n - \frac{n^2+2n+2}{2}) + \frac{n^2}{2}(e^n - \frac{n^2+2n+2}{2}) + 1 - e^n\left(\frac{n^2-2n+2}{2}\right) - e^{-n}\left(\frac{n^2+2n+2}{2}\right) + \frac{1}{4}(n^4 - 2n^3 + 2n^2 + 2n^3 - 4n^2 + 4n + 2n^2 - 4n + 4)\}]$

$= [I + B^3\{e^{-n} - \frac{n^2-2n+2}{2} + \frac{n^3}{2} + n(e^{-n} - \frac{n^2-2n+2}{2}) - \frac{n^3}{2} + \frac{n^4}{4} + \frac{n^3}{2}(e^{-n} - \frac{n^2-2n+2}{2}) + (e^n - \frac{n^2+2n+2}{2}) - n(e^n - \frac{n^2+2n+2}{2}) + \frac{n^2}{2}(e^n - \frac{n^2+2n+2}{2}) + 1 - e^n\left(\frac{n^2-2n+2}{2}\right) - e^{-n}\left(\frac{n^2+2n+2}{2}\right) + \frac{1}{4}(n^4+4)\}]$

= I, after simplification.

Theorem(2.II):-

$e^{nB} \times e^{mB} = e^{(n+m)B}$

Proof: Let $e^{nB} = [I + nB + \frac{n^2}{2}B^2 + B^3\{e^n - \frac{n^2+2n+2}{2})]$

And $e^{mB} = [I + mB + \frac{m^2}{2}B^2 + B^3\{e^m - \frac{m^2+2m+2}{2})]$

Then, $e^{nB} \times e^{mB}$

$= [I + nB + \frac{n^2}{2}B^2 + B^3\{e^n - \frac{n^2+2n+2}{2})]\ [I + mB + \frac{m^2}{2}B^2 + B^3 e^m - \frac{m^2+2m+2}{2})]$

$= I + mB + \frac{m^2}{2}B^2 + B^3\{e^m - \frac{m^2+2m+2}{2}) + nB + nmB^2 + \frac{nm^2}{2}B^3$
$+ nB^4(e^m - \frac{m^2+2m+2}{2}) + \frac{n^2}{2}B^2 + \frac{mn^2}{2}B^3 + \frac{n^2m^2}{4}B^4 + \frac{n^2}{2}B^3(e^m - \frac{m^2+2m+2}{2})$
$+ B^3(e^n - \frac{n^2+2n+2}{2}) + mB^4(e^n - \frac{n^2+2n+2}{2}) + \frac{m^2}{2}B^5(e^n - \frac{n^2+2n+2}{2})$
$+ B^6(e^n - \frac{n^2+2n+2}{2})(e^m - \frac{m^2+2m+2}{2})]$

$= [I + (n + m)B + \frac{(n+m)^2}{2}B^2 + B^3(e^{n+m} - nm - n - m - \frac{n^2}{2} - \frac{m^2}{2} - 1)]$

$= [I + (n + m)B + \frac{(n+m)^2}{2}B^2 + B^3(e^{n+m} - \frac{n^2+m^2+2nm+2n+2m+2}{2})]$

$= [I + (n + m)B + \frac{(n+m)^2}{2}B^2 + B^3(e^{n+m} - \frac{(n+m)^2+2(n+m)+2}{2})]$

i.e. $e^{nB} \times e^{mB} = e^{(n+m)B}$

Theorem(2.III):-

If A and B are two (4,3)-jection then $e^A e^B = e^{A+B}$, when AB = BA= 0

Proof: Let A and B (4,3)- jections, so that

e^B = [I + B + $\frac{B^2}{2!}$ + B^3 {e $-\frac{5}{2}$}]
e^A = [I + A + $\frac{A^2}{2!}$ + A^3 {e $-\frac{5}{2}$}]
Then we find that,
$e^A e^B$ = [I + A + $\frac{A^2}{2!}$ +A^3 {e $-\frac{5}{2}$}] [I + B + $\frac{B^2}{2!}$ +B^3{e $-\frac{5}{2}$}]
= [I + B + $\frac{B^2}{2!}$ +B^3 {e $-\frac{5}{2}$}+ A + AB + $\frac{AB^2}{2} + AB^3(e-\frac{5}{2}) + \frac{A^2}{2} + \frac{A^2B}{2} + \frac{A^2B^2}{4} + \frac{A^2B^3}{2}(e-\frac{5}{2}) + A^3(e-\frac{5}{2}) + A^3B(e-\frac{5}{2}) + \frac{B^2A^3}{2}(e-\frac{5}{2}) + A^3(e-\frac{5}{2})B^3(e-\frac{5}{2})$]
e^{A+B}= [I + A + B + $\frac{A^2}{2} + \frac{B^2}{2!} + \frac{A^2B^2}{4} + A^3(e-\frac{5}{2}) + B^3$ {e $-\frac{5}{2}$} + AB + $\frac{AB^2}{2}$ + A$B^3(e-\frac{5}{2}) + \frac{A^2B}{2} + \frac{A^2B^3}{2}(e-\frac{5}{2}) + A^3B(e-\frac{5}{2}) + \frac{B^2A^3}{2}(e-\frac{5}{2}) + A^3(e-\frac{5}{2})B^3(e-\frac{5}{2})$]
We need A + B to be a (4,3)− jection
i.e. $(A+B)^4 = (A+B)^3$
$A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4 = A^3 + 3A^2B + 3B^2A + B^3$
Assuming A, B commute this possible when AB = 0 = BA
Then $e^A e^B = e^{A+B}$

3. Projection operator

Definitions:

Let L be a linear space. Let B be a projection on L. then B is a linear transformation from L into L such that $B^2 = B$

Define, e^B= I + B + $\frac{B^2}{2!} + \frac{B^3}{3!} + \frac{B^4}{4!} + \frac{B^5}{5!} + \frac{B^6}{6!}$ + --------

= I + B + $\frac{B}{2!} + \frac{B}{3!} + \frac{B}{4!} + \frac{B}{5!} + \frac{B}{6!}$ + --------

= I + B(1 + $\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$ + ------------)

= I + B(e −1) ---(3.1)

Theroem(3.I):

If B is a (4,3)−jection then B is also a projection.

Proof :- Let B is a (4,3)-jection, then we have

e^B = [I + B + $\frac{B^2}{2!}$ + B^3 {e $-\frac{5}{2}$}]
= [I + B + $\frac{B}{2!}$ + B {e $-\frac{5}{2}$}], if B^2 = B,
= I + B(e – 1) from (3.1) which show that B is also a projection
e^B = I + B(e – 1) ; it is a projection.

Theorem (3.II):

If B is a (4,3)−jection, then B^2 is also a projection .

Proof :- Let B is a (4,3)-jection, then we have

e^B = [I + B + $\frac{B^2}{2!}$ +B^3 {e $-\frac{5}{2}$}]
e^{B^2}= [I + B^2 + $\frac{(B^2)^2}{2!}$ +$(B^2)^3$ {e $-\frac{5}{2}$}]
= [I + B^2 + $\frac{B^4}{2!}$ +B^6 {e $-\frac{5}{2}$}], from (3.1) as B^2 = B then
= [I + B + $\frac{B}{2!}$ + B {e $-\frac{5}{2}$}]
e^{B^2}= I + B(e – 1); it is a projection.

Theorem(3.III):

If B is a (4,3)−jection, then (I $- B^2$) is also a projection.

Proof:- Let B is a (4,3)-jection, then we have

e^B = [I + B + $\frac{B^2}{2!}$ + B^3 {e $-\frac{5}{2}$}]

$e^{\mathrm{I}-B^2}$ = [I + (I − B^2) + $\frac{(\mathrm{I}-B^2)^2}{2!}$ + (I − $B^2)^3$ {e − $\frac{5}{2}$}]

= I + (I − B^2) + $\frac{I^2+B^4-2B^2I}{2}$ + $(I^3 - B^6 - 3I^2B^2 + 3B^4I)\{e - \frac{5}{2}\}$From (3.1) as B^2 = B then $e^{(\mathrm{I}-B^2)}$ = I + B(e − 1); it is a projection.

4. Definition of a hyperbolic function of (4,3)-jection operator and some results.

$\cos hB = \mathrm{I} + \frac{B^2}{2!} + \frac{B^4}{4!} + \frac{B^6}{6!}$-------------------------------------; (if B^4=B^3)

= I + $\frac{B^2}{2!} + \frac{B^3}{4!} + \frac{B^3}{6!}$ -------------------------------------

= I + $\frac{B^2}{2!} + B^3[\frac{1}{4!} + \frac{1}{6!}$ --]

= I + $\frac{B^2}{2!} + B^3[(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!}$ -------------------------------------) − (1+ $\frac{1}{2!}$)]

= I + $\frac{B^2}{2!} + B^3[\frac{e+e^{-1}}{2} - \frac{3}{2}]$

$\cos hB$ = I + $\frac{B^2}{2!} + B^3[\cos h1 - \frac{3}{2}]$

And $\sin hB = B + \frac{B^3}{3!} + \frac{B^5}{5!} + \frac{B^7}{7!}$ ---------------------------------; (if B^4= B^3)

$\sin hB = B + \frac{B^3}{3!} + \frac{B^3}{5!} + \frac{B^3}{7!}$ ---

= $B + B^3[\frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!}$ ---]

= $B + B^3[(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!}$ ---) −1]

= B + $B^3[\frac{e-e^{-1}}{2} - 1]$

$\sin hB = B + B^3[\sin h1 - 1]$

When B is projection; i.e. B^2= B

Then, $\cos hB$ = [I + $\frac{B^2}{2!} + B^3[\cos h1 - \frac{3}{2}]$; ($B^2$ = B)

= I + $\frac{B}{2!}$+ $B[\cos h1 - \frac{3}{2}]$

$\cos hB$ = I + $B[\cos h1 - 1]$

And, $\sin hB = B + B^3[\sin h1 - 1]$; ($B^2$ = B)

= $B + B[\sin h1 - 1]$

$\sin hB = B \sin h1$

$\sin hB + \cos hB$

= $[B + B^3\{\sin h1 - 1\}]$ + [I + $\frac{B^2}{2!}$+ $B^3\{\cos h1 - \frac{3}{2}\}$]

= [I + B + $\frac{B^2}{2!} + B^3\{\sin h1 - 1 + \cos h1 - \frac{3}{2}\}$]

$\sin hB + \cos hB$ = [I + B + $\frac{B^2}{2!} + B^3(e - \frac{5}{2})$]---(4.1)

$\cos hB - \sin hB$

= [I + $\frac{B^2}{2!}$+ $B^3\{\cos h1 - \frac{3}{2}\}] - [B + B^3\{\sin h1 - 1\}]$

= [I + $\frac{B^2}{2!}$+ $B^3\{\cos h1 - \frac{3}{2}\} - B - B^3\{\sin h1 - 1\}$]

= [I $-B$ + $\frac{B^2}{2!}$+ $B^3\{\cos h1 - \frac{3}{2} - \sin h1 + 1\}$]

= [I $-B$ + $\frac{B^2}{2!}$+ $B^3\{\frac{e+e^{-1}}{2} - \frac{e-e^{-1}}{2} - \frac{1}{2}$]

= [I $- B + \frac{B^2}{2!}$ + $B^3\{ e^{-1} - \frac{1}{2}$]

$\cos hB - \sin hB$ = [I $-B + \frac{B^2}{2!}$+ $B^3\{ e^{-1} - \frac{1}{2}$]---------------------------------(4.2)

Theorem (4.I):- $\cos h^2 B - \sin h^2 B = I$

Proof : L.H.S

$\cos h^2 B - \sin h^2 B$

= [$\sin hB$ + $\cos hB$][$\cos hB - \sin hB$] (from 4.1 and 4.2)

= [I + B + $\frac{B^2}{2!} + B^3(e - \frac{5}{2})$] [I $-B + \frac{B^2}{2!}$+ $B^3\{ e^{-1} - \frac{1}{2}\}$]

= [I $-B + \frac{B^2}{2!} + B^3\{ e^{-1} - \frac{1}{2}\}$ + B $- B^2 + \frac{B^3}{2} + B^4\{ e^{-1} - \frac{1}{2}\} + \frac{B^2}{2} - \frac{B^3}{2} + \frac{B^4}{4} + \frac{B^5}{2}\{ e^{-1} - \frac{1}{2}\}$ + $B^3\left(e - \frac{5}{2}\right) - B^4\left(e - \frac{5}{2}\right) + \frac{B^5}{2}\left(e - \frac{5}{2}\right) + B^6\left(e - \frac{5}{2}\right)\{ e^{-1} - \frac{1}{2}\}$] (if $B^4 = B^3$)

= [I $-B + \frac{B^2}{2!} + B^3\{ e^{-1} - \frac{1}{2}\}$ + B $- B^2 + \frac{B^3}{2} + B^3\{ e^{-1} - \frac{1}{2}\} + \frac{B^2}{2} - \frac{B^3}{2} + \frac{B^3}{4} + \frac{B^3}{2}\{ e^{-1} - \frac{1}{2}\}$ + $B^3\left(e - \frac{5}{2}\right) - B^3\left(e - \frac{5}{2}\right) + \frac{B^3}{2}\left(e - \frac{5}{2}\right) + B^3\left(e.e^{-1} - \frac{e^{-1}}{2} - \frac{5e^{-1}}{2} + \frac{5}{4}\right)$]

= [I $-B^3(0)$]

Hence; $\cos h^2 B - \sin h^2 B$ = I

Adding (2.2) and (2.3) we get

$e^{nB} + e^{-nB} = [2I + 2.\frac{n^2}{2}B^2 + B^3\{e^n + e^{-n} - \frac{n^2 + 2n + 2}{2} - \frac{n^2 - 2n + 2}{2}\}]$

$= [2I + 2.\frac{n^2}{2}B^2 + B^3\{e^n + e^{-n} - (\frac{n^2 + 2n + 2 + n^2 - 2n + 2}{2}\}]$

$2\cos hnB = [2I + 2.\frac{n^2}{2}B^2 + B^3\{2\cos hn - (\frac{2n^2 + 4}{2}\}]$

$\cos hnB = [I + \frac{n^2}{2}B^2 + B^3\{\cos hn - (\frac{n^2 + 2}{2})\}]$---------------------------------(4.3)

Subtract (2.2) and (2.3) we get

$e^{nB} - e^{-nB} = [I + nB + \frac{n^2}{2}B^2 + B^3\{e^n - \frac{n^2 + 2n + 2}{2}\}] - [I - nB + \frac{n^2}{2}B^2 + B^3\{e^{-n} - \frac{n^2 - 2n + 2}{2}\}]$

$e^{nB} - e^{-nB} = [2nB + B^3\{e^n - e^{-n} - \frac{n^2 + 2n + 2}{2} + \frac{n^2 - 2n + 2}{2}\}]$

$e^{nB} - e^{-nB} = [2nB + B^3\{e^n - e^{-n} - \frac{n^2 + 2n + 2 - n^2 + 2n - 2}{2}\}]$

$e^{nB} - e^{-nB} = [2nB + B^3\{e^n - e^{-n} - \frac{4n}{2}\}]$

$2\sin hnB = [2nB + B^3\{2\sin hn - 2n\}]$

$\sin hnB = [nB + B^3\{\sin hn - n\}]$---------------------------------(4.4)

Now, we establish some formulae analogous to trigonometry

Putting n = -1 in (4.3) we get

$\cos h(-1)B = [I + \frac{(-1)^2}{2}B^2 + B^3\{\cos h(-1) - (\frac{(-1)^2 + 2}{2}\}]$

$\cos hB = [I + \frac{1}{2}B^2 + B^3\{\cos h - \frac{3}{2}\}]$

i.e. $\cos h(-1)B = \cos hB$

Again putting n = -1 in (4.4)

$\sin h(-1)B = [(-1)B + B^3\{\sin h(-1) - (-1)\}]$

$= [-B + B^3\{-\sin h1 + 1\}]$

$= -[B + B^3\{\sin h1 - 1\}]$

$\sin h(-1)B = -\sin hB$

Now putting n = 2 in (4.4) we get

$\sin h2B = [2B + B^3\{\sin h2 - 2\}]$

$\sin h2B = [2B + B^3\{2\sin h1 \cos h1 - 2\}]$ ---------------------------(4.5)

Again putting n = 2 in (4.3)

$\cos h2B = [I + \frac{(2)^2}{2}B^2 + B^3\{\cos h2 - (\frac{2^2 + 2}{2})\}]$

$\cos h2B = [I + 2B^2 + B^3\{\cos h2 - 3\}]$ ---------------------------------(4.6)

Theorem (4.II):- $\sin h2B = 2\sin hB \cos hB$

Proof :- R.H.S

$2\sin hB \cos hB$

= 2 $[B + B^3(\sin h1 - 1)]$ [I + $\frac{B^2}{2!}$+ $B^3(\cos h1 - \frac{3}{2})$]

= 2 $[B + B^3(\sin h1 - 1) + \frac{B^3}{2!} + \frac{B^5}{2!}(\sin h1 - 1) + B^4(\cos h1 - \frac{3}{2}) + B^6(\sin h1 - 1)(\cos h1 - \frac{3}{2})$ (if $B^4 = B^3$)

= 2 $[B + B^3(\sin h1 - 1) + \frac{B^3}{2!} + \frac{B^3}{2!}(\sin h1 - 1) + B^3(\cos h1 - \frac{3}{2}) + B^3(\sin h1 \cos h1 - \frac{3}{2}\sin h1 - \cos h1 + \frac{3}{2})]$

= 2 $[B + B^3\{\sin h1 \cos h1 - 1\}]$

$2\sin hB \ \cos hB$ = $[2B + B^3\{2\sin h1 \cos h1 - 2\}]$ from (4.3)

$\sin h2B = 2\sin hB \ \cos hB$

Theorem (4.III):- $\cos h2B = \cos h^2B + \sin h^2B$

Proof :- R.H.S

$\cos h^2B + \sin h^2B$

= [I + $\frac{B^2}{2!}$ + $B^3[\cos h1 - \frac{3}{2}]^2$ + $[B + B^3(\sin h1 - 1)]^2$

= $I^2 + (\frac{B^2}{2!})^2 + \{B^3(\cos h1 - \frac{3}{2})\}^2$+ 2.I.$\frac{B^2}{2!}$ + 2.$\frac{B^2}{2!}$.$B^3(\cos h1 - \frac{3}{2})$ +2.I.$B^3(\cos h1 - \frac{3}{2})$ + B^2 + 2.B. $B^3(\sin h1 - 1)]$ + $\{B^3(\sin h1 - 1)\}^2$

= I + $\frac{B^4}{4}$ + $B^6(\cos h1 - \frac{3}{2})^2$+ 2.I.$\frac{B^2}{2!}$ + 2.$\frac{B^5}{2!}$. $(\cos h1 - \frac{3}{2})$ +2.$B^3(\cos h1 - \frac{3}{2})$ + B^2+ 2. $B^4(\sin h1 - 1)]$ + $B^6(\sin h1 - 1)^2$(if $B^4 = B^3$)

= I + $\frac{B^3}{4}$ + $B^3(\cos h^2 1 + \frac{9}{4} - 3\cos h1)$+ B^2 + $B^3(\cos h1 - \frac{3}{2})$ +$2B^3 \cos h1 - 3B^3$ + B^2 + 2. $B^3 \sin h1 - 2.B^3$ +B^3 ($\sin h^2 1 + 1 - 2\sin h1$)

= I + $2B^2$ + B^3($\cos h^2 1$+ $\sin h^2 1 - 3$)

= I + $2B^2$ + B^3($\cos h2 - 3$) from (4.4)

= $\cos h2B$

Hence, $\cos h2B = \cos h^2B + \sin h^2B$

5. Sine and cosine function of opeators :-

Next we define sine and cosine functions

Let $\sin nB = \frac{1}{i}\sin hinB$

= $\frac{1}{i}[inB + B^3(\sin hin - in)]$

= $\frac{1}{i}[inB + B^3(\mathrm{i}\sin n - in)]$

= $[nB + B^3(\sin n - n)]$

$\sin nB$ = $[nB + B^3(\sin n - n)]$--(5.1)

And, $\cos nB$ = $\cos hinB$

$\cos nB$ = [I + $\frac{(in)^2}{2}B^2$ + $B^3\{\cos hin - (\frac{(in)^2+2}{2}\}$]

= [I + $\frac{-(n)^2}{2}B^2$ + $B^3\{\cos n - (\frac{-(n)^2+2}{2}\}$]

$\cos nB$ =[I $- \frac{(n)^2}{2}B^2$ + $B^3\{\cos n + \frac{(n)^2-2}{2}\}$] -----------------------------(5.2)

If B is projection then,

$\sin nB$ = $[nB + B(\sin n - n)]$ (if $B^2 = B$)

$\sin nB$ = $B \sin n$

$\cos nB$ = [I $- \frac{(n)^2}{2}B$ + B$\{\cos n + \frac{(n)^2-2}{2}\}$(if $B^2 = B$)

= [I $- \frac{(n)^2}{2}B$ + B$\cos n$ + $\frac{(n)^2}{2}B - B$]

= [I + B $(\cos n - 1)$]

$\cos nB$ = [I + B $(\cos n - 1)$]

Putting, n = 1 in (5.1) and (5.2)

$\sin B1$ = $[1B + B^3(\sin 1 - 1)]$

$\sin B$ = $[B + B^3(\sin 1 - 1)]$

And, $\cos B1 = [\mathrm{I} - \frac{(1)^2}{2} B^2 + B^3\{\cos 1 + \frac{(1)^2 - 2}{2}\}]$
$\cos B = [\mathrm{I} - \frac{1}{2} B^2 + B^3\{\cos 1 - \frac{1}{2}\}]$
Putting, n = 2 in (5.1) and (5.2)
$\sin 2B = [2B + B^3(\sin 2 - 2)]$

$\cos 2B = [\mathrm{I} - \frac{(2)^2}{2} B^2 + B^3\{\cos 2 + \frac{(2)^2 - 2}{2}\}]$

$\cos 2B = [\mathrm{I} - 2B^2 + B^3\{\cos 2 + 1\}]$

Now, $e^{iB} = \mathrm{I} + \mathrm{i}B - \frac{B^2}{2!} - \mathrm{i}\frac{B^3}{3!} + \frac{B^4}{4!} + i\frac{B^5}{5!} - \frac{B^6}{6!} + \text{--------}$ (if $B^4 = B^3$)

$e^{iB} = \mathrm{I} + \mathrm{i}B - \frac{B^2}{2!} + B^3(e^i - i - \frac{1}{2})$

$= \mathrm{I} + \mathrm{i}B - \frac{B^2}{2!} + B^3(\cos 1 + i\sin 1 - i - \frac{1}{2})$
$= [\mathrm{I} - \frac{B^2}{2!} + B^3(\cos 1 - \frac{1}{2}) + \mathrm{i}B + \mathrm{B}^3\ (i\sin 1 - i)]$
$= [\mathrm{I} - \frac{B^2}{2!} + B^3(\cos 1 - \frac{1}{2})] + i[\mathrm{B} + \mathrm{B}^3\ (\sin 1 - 1)]$
$e^{iB} = \cos B + \mathrm{i}\sin B$
We establish formulae analogous to trigonometry.
Theorem (5.I):- $(\cos B)^2 + (\sin B)^2 = \mathrm{I}$
Proof :- L.H.S $(\cos B)^2 + (\sin B)^2$
Using relations, $[\mathrm{I} - \frac{1}{2}B^2 + B^3\{\cos 1 - \frac{1}{2}\}]^2 + [B + B^3(\sin 1 - 1)]^2$
$= [\mathrm{I}^2 + (-\frac{1}{2}B^2)^2 + \{B^3(\cos 1 - \frac{1}{2})\}^2 + 2.\mathrm{I}\,(-\frac{1}{2}B^2) - 2.\frac{1}{2}B^2 B^3(\cos 1 - \frac{1}{2}) + 2.\mathrm{I}\ B^3(\cos 1 - \frac{1}{2}) + B^2 + \{B^3(\sin 1 - 1)\}^2 + 2.\mathrm{B}.B^3(\sin 1 - 1)]$
$= [\mathrm{I} + \frac{1}{4}B^4 + B^6\{(\cos 1)^2 + \frac{1}{4} - 2\cos 1\} - 2.\frac{1}{2}B^2) - 2.\frac{1}{2}B^5(\cos 1 - \frac{1}{2}) + 2\ B^3(\cos 1 - \frac{1}{2}) + B^2 + B^6\{(\sin 1)^2 + 1 - 2\sin 1\} + 2B^4(\sin 1 - 1)]$(if $B^4 = B^3$)
$= [\mathrm{I} + \frac{1}{4}B^3 + B^3\{(\cos 1)^2 + \frac{1}{4} - 2\cos 1\} - 2.\frac{1}{2}B^2) - 2.\frac{1}{2}B^3(\cos 1 - \frac{1}{2}) + 2\ B^3(\cos 1 - \frac{1}{2}) + B^2 + B^3\{(\sin 1)^2 + 1 - 2\sin 1\} + 2B^3(\sin 1 - 1)]$
$= [\mathrm{I} + B^3\{(\cos 1)^2 + (\sin 1)^2 - 1\}]$
$= [\mathrm{I} + \mathrm{B}^3\{0\}]$
Hence, $(\cos B)^2 + (\sin B)^2 = \mathrm{I}$
Theorem(5.II):- $\cos 2B = (\cos B)^2 - (\sin B)^2$
Proof :- R.H.S
$(\cos B)^2 - (\sin B)^2$
$= [\sin B + \cos B][\cos B - \sin B]$
$= [\{\mathrm{I} - \frac{1}{2}B^2 + B^3(\cos 1 - \frac{1}{2})\} + \{B + B^3(\sin 1 - 1)\}]\ [\{\mathrm{I} - \frac{1}{2}B^2 + B^3(\cos 1 - \frac{1}{2})\} - \{B + B^3(\sin 1 - 1)\}]$
$= [\mathrm{I} - \frac{1}{2}B^2 + B^3(\cos 1 - \frac{1}{2}) + B + B^3(\sin 1 - 1)]\ [\mathrm{I} - \frac{1}{2}B^2 + B^3(\cos 1 - \frac{1}{2}) - B - B^3(\sin 1 - 1)\}]$
$= [\mathrm{I} + B - \frac{1}{2}B^2 + B^3\{\cos 1 + \sin 1 - \frac{3}{2})\}]\ [\mathrm{I} - B - \frac{1}{2}B^2 + B^3\{\cos 1 - \sin 1 + \frac{1}{2})\}]$
$= \mathrm{I} - B - \frac{1}{2}B^2 + B^3\{\cos 1 - \sin 1 + \frac{1}{2})\} + \mathrm{B} - B^2 - \frac{1}{2}B^3 + B^4\{\cos 1 - \sin 1 + \frac{1}{2})\} - \frac{1}{2}B^2 + \frac{1}{2}B^3 + \frac{1}{4}B^4 - \frac{1}{2}B^5\{\cos 1 - \sin 1 + \frac{1}{2})\} + B^3\{\cos 1 + \sin 1 - \frac{3}{2}) - B^4\{\cos 1 + \sin 1 - \frac{3}{2}) - \frac{1}{2}B^5\{\cos 1 + \sin 1 - \frac{3}{2}) + B^6\{\cos 1 + \sin 1 - \frac{3}{2}\}\{\cos 1 - \sin 1 + \frac{1}{2})\}$ (if $B^4 = B^3$)
$= \mathrm{I} - B - \frac{1}{2}B^2 + B^3\{\cos 1 - \sin 1 + \frac{1}{2})\} + \mathrm{B} - B^2 - \frac{1}{2}B^3 + B^3\{\cos 1 - \sin 1 + \frac{1}{2})\} - \frac{1}{2}B^2 + \frac{1}{2}B^3 + \frac{1}{4}B^3 - \frac{1}{2}B^3\{\cos 1 - \sin 1 + \frac{1}{2})\} + B^3\{\cos 1 + \sin 1 - \frac{3}{2}) - B^3\{\cos 1 + \sin 1 - \frac{3}{2}) - \frac{1}{2}B^3\{\cos 1 + \sin 1 - \frac{3}{2}) + B^3\{\cos 1 + \sin 1 - \frac{3}{2}\}\{\cos 1 - \sin 1 + \frac{1}{2})\}$
$= \mathrm{I} - 2B^2 + B^3\{(\cos 1)^2 - (\sin 1)^2 + 1\}$

$= I - 2B^2 + B^3 \{\cos 2 + 1\}$
$= \cos 2B$
<u>Theorem (5.III)</u>:-$\sin 2B = 2 \sin B \cos B$
Proof :- R.H.S
$2 \sin B \cos B$
$= 2 [B + B^3(\sin 1 - 1)] [I - \frac{1}{2}B^2 + B^3(\cos 1 - \frac{1}{2})]$
$= 2 [B - \frac{1}{2}B^3 + B^4(\cos 1 - \frac{1}{2}) + B^3(\sin 1 - 1) - \frac{1}{2}B^5(\sin 1 - 1) + B^6(\sin 1 - 1)(\cos 1 - \frac{1}{2})]$
(if $B^4 = B^3$)
$= 2 [B - \frac{1}{2}B^3 + B^3(\cos 1 - \frac{1}{2}) + B^3(\sin 1 - 1) - \frac{1}{2}B^3(\sin 1 - 1) + B^3(\sin 1 \cos 1 - \cos 1 - \frac{1}{2}\sin 1 + \frac{1}{2})]$
$= 2 [B + B^3(\sin 1 \cos 1 - 1)]$
$= [2B + B^3(2\sin 1 \cos 1 - 2)]$
$= [2B + B^3(\sin 2 - 2)]$
$= \sin 2B$

6. Differentiation of operators:-

Now we are going to introduce differentiation in respect of e^{xB} with respect to x. we generalise formulae for e^{nB} to e^{xB}, x being any real number by

$e^{xB} = [I + xB + \frac{x^2}{2}B^2 + B^3\{e^x - \frac{x^2 + 2x + 2}{2}\}]$

Theorem (6.I):-$\frac{d}{dx}(e^{xB}) = Be^{xB}$

Proof:-$\frac{d}{dx}(e^{xB}) = \lim_{\Delta x \to 0} \frac{e^{(x + \Delta x)B} - e^{xB}}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{e^{(xB + \Delta xB)} - e^{xB}}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{xB}.e^{\Delta xB} - e^{xB}}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{e^{xB}(e^{\Delta xB} - 1)}{\Delta x} = e^{xB} \lim_{\Delta x \to 0} \frac{(e^{\Delta xB} - 1)}{\Delta x}$

$= e^{xB} \lim_{\Delta x \to 0} \frac{[I + \Delta xB + \frac{(\Delta x)^2 B^2}{2} + B^3\{e^{\Delta x} - \frac{(\Delta x)^2 + 2\Delta x + 2}{2}\} - I]}{\Delta x}$

$= e^{xB} \lim_{\Delta x \to 0} \frac{[\Delta xB + \frac{(\Delta x)^2 B^2}{2} + B^3\{I + \Delta x + \frac{(\Delta x)^2}{2!} + \frac{(\Delta x)^3}{3!} ------- - \frac{(\Delta x)^2}{2!} - \Delta x - 1\}]}{\Delta x}$

$= e^{xB} \lim_{\Delta x \to 0} \frac{[\Delta xB + \frac{(\Delta x)^2 B^2}{2} + B^3\{\frac{(\Delta x)^3}{3!} + \frac{(\Delta x)^4}{4!} ---------\}]}{\Delta x}$

$= e^{xB} \lim_{\Delta x \to 0} [B + \frac{(\Delta x)B^2}{2} + B^3 \{\frac{(\Delta x)^2}{3!} + \frac{(\Delta x)^3}{4!} \text{ higher power of } \Delta x\}]$

$= B\, e^{xB}$

Also

$\frac{d}{dx}(e^{-xB}) = \frac{d}{d(-x)}(e^{-xB})\frac{d}{dx}(-x) = -Be^{-xB}$

So $\frac{d}{dx}(\frac{e^{xB} + e^{-xB}}{2}) = (\frac{Be^{xB} - Be^{-xB}}{2}) = B(\frac{e^{xB} - e^{-xB}}{2})$

$= B \sin hxB$

Thus,$\frac{d}{dx}(\cos hxB) = B \sin hxB$

Similarly, $\frac{d}{dx}(\sin hxB) = B \cos hxB$

So, $\frac{d}{dx}(\cos hxB) = B\{Bx + B^3(\sin hx - x)\}$
$= \{B^2 x + B^4(\sin hx - x)\}$
$= \{B^2 x + B^3(\sin hx - x)\}$(if $B^4 = B^3$)

$\frac{d}{dx}(\cos hxB) = B^2\{x + B(\sin hx - x)\}$

$\frac{d}{dx}(\sin hxB) = B[I + \frac{B^2 x}{2!} + B^3(\cos hx - \frac{3}{2}x)]$

$= [B + \frac{B^3 x}{2!} + B^4(\cos hx - \frac{3}{2}x)]$

$= [B + \frac{B^3 x}{2!} + B^3(\cos hx - \frac{3}{2}x)]$ (if $B^4 = B^3$)

$\frac{d}{dx}(\sin hxB)$ = $B[I + \frac{B^2 x}{2!} + B^2(\cos hx - \frac{3}{2}x)]$

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