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S – CONVEX SET IN A TOPOLOGICAL VECTOR SPACE

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Abstract: In this present article, we explore S-Convex Sets in topological vector spaces, investigating key concepts like closed sets, open sets, set interiors, and the closure property of S-Convex Sets. It establishes that the closure of an S-Convex Set in a topological vector space remains S-Convex, which differs in technique from metric spaces. Neighborhood systems are employed. It presents theorems involving closed sets, set closures under specific conditions, and the interior of sets in topological vector spaces. It demonstrates that the interior of an S-Convex Set in such a space also preserves the S-Convex property. Furthermore, the chapter derives a result related to S-Convex Sets and S-Convex hulls.

Keywords: Open set, convex set, closure, topological vector spaces.

Introduction

In the real life problem of optimization, economics, engineering, convex sets are fundamental. A convex set is a set that exhibits a particular geometric property, and understanding this property is essential for a wide range of theoretical and practical problems. This introduction provides an overview of convex sets and offers a concise literature survey to highlight their importance and applications.

In Euclidean space, a convex set is one that has every line segment between any two locations in the set. Formally speaking, a set S is convex if every pair of points x and y in S has a line segment between x and y that lies wholly within S. This geometric definition of convexity serves as the foundation for a wide range of theoretical and applied mathematics.

The first study according to our knowledge is done by Minkowski in [8]. Early developments of convexity were finite dimensional and concerned with solution of quantitative problems, which has been described in [4].Understanding convex sets is pivotal as they serve as a fundamental building block for convex optimization, which, in turn, is a powerful tool in solving various real-world problems. In [12], the extreme points and related results. Convex sets underpin the efficient algorithms used in resource allocation, portfolio optimization, image reconstruction, and many other fields. Their geometric properties ensure that optimization problems associated with convex sets have unique solutions, making them an indispensable concept in mathematical modeling. In [9], the authors introduce a convex-constrained image restoration model that effectively improves image quality by optimizing the relaxation. We can see the literature regarding the properties of convex sets in [1-3, 5, 6, 7, 10, 11, 13] and references therein and their applications.

In this present paper, our focus has been on the in-depth exploration of S-convex sets, a concept with profound mathematical implications. We have not only introduced the idea of S-convex sets but have gone a step further to define topological operations on S-convex sets.

Our work efforts have been on the study of S-Convex Set in a topological vector space. We have also studied the notions of a closed set, open set, interior of a set, and the closure property of S-Convex Sets and using the notions of these we have established a few results. We have observed that in a topological vector space the closure of a S-Convex Set is again a S–convex set since such result also holds good in a metric space and hence in a normed linear space because a metric space is also a normed linear space. But while we were establishing this result in a topological vector space, we observed that the technique is not analogous to that for metric spaces. Also, in proving the result we have adopted the neighborhood system in place of open (or closed) sphere system. We have also used the results given in the section for essential definitions in establishing the results. One more theorem we have established in the 3rd section of this chapter using the notion of the closed set and the closure of a set under a set of suitable conditions. In addition to that we have also established a theorem using the notion of the interior of a set in a topological vector space. Through this theorem we have observed that the interior of a S-Convex Set in a topological vector space is again a S-Convex Set. The property of an interior of a set A that it cannot be a single ton set worked a lot in getting two elements in A with suitable scalars to from A, a S-Convex Set.

We have also established a result using the notion of S-Convex Set and S-convex hull in a topological vector space. In this theorem we have observed that the S-convex hull of an open set is open. In establishing this theorem, we used the property of S-Convex Hull that an element of it can be expressed in terms of finite S-linear combinations. Another thing on which we have made our base to establish this result is the use of the notion of neighborhood system.

One more thing which we would like to bring in the notice that there is a scope of huge work in topological vector space with the notion of S – convex sets. Also, one can go ahead with the notion of Balanced set, absorbing set etc. Also, a good number of results can be obtained with the notion of balanced set analogous to that for S-Convex Set.

The present article is divided in two sections. In the next section, we give all the required definitions. In last section, we give detailed proof of our results.

Preliminaries and Definitions

In this section, we present essential preliminaries and definitions which are useful in establishing some of the results.

Definition1:Metric (or Distance Function) Let M be a non-empty set then a real valued function d defined on $M \times M$ is called a distance function (or metric function or simply metric on M) if the following conditions are satisfied:

i) $d(x, y) \geq 0$

ii) $d(x, y) = 0 \Leftrightarrow x = y$

iii) $d(x, y) = d(y, x)$

iv) $d(x, z) \leq d(x, y) - d(y, z)$.

Here the condition (iii) is known as the condition of symmetry and condition (iv) is known as triangle inequality.

Here (x, y) is called the distance between x and y.

Also, d (x, y) due to symmetry does not depend on the order of the element.

Definition 2: Metric Space The system (or the pair) (M, d) containing nonempty set M and a metric d defined on it is caused a metric space. The elements of M are called the points of the metric space (M, d). *[we refer to Simmons, G. F. (1) P. 51]*

Definition 3: Topology Let X be a non-empty set. A class T of subsets of X is called a topology on X if it satisfies the following two conditions:

1) The union of every class of sets in T is a set in T.
2) The intersection of every finite class of sets in T is a set in T.

That is a topology on a given nonempty set X is a class of X which is closed under the formation of arbitrary unions and finite intersections.

Definition 4: Topological Space A topological space (X, T) is a system consisting of a non-empty set X and a topology T defined on X. The sets in the class T are called the open sets of the topological space (X, T), and elements of X are called its points. No harm can come from this practice that if one, instead of writing (X, T), simply writes X.[*We refer to Simmons, G. F. (1) P. 92]*.

Definition 5: Open set A subset G of the metric space X is called an open set if, given any point x in G, there exists a positive real number r such that $S_r(x) \subseteq G$.

That is, if each point of G is the center of some open sphere contained in G.

Also, on the real line, a set consisting of a single point is not open, for each bounded open interval centered on the point contains points not in the set.

Result 1: In any metric space X, the empty set $\varnothing$ and the full space X are open sets. Also, each open sphere in X is an open set.

[For the definition and the proof of the result 4 we refer to Simmons, G. F. (1) P. 60]

Definition 6: Closed set Let (X, T) be a topological space. A subset A of X is said to be closed set if its compliments A^c is an open set.

That is if, $X - A = A^c \in T$.

We now give same result on it which we shall use in the next section to establish some of the results.

***Result 2**:* $\varnothing$ and X are closed sets.

***Result 3**:* Any arbitrary intersection of closed sets in X is a closed set in X.

***Result 4**:* The union of any finite number of closed sets in X is a closed set is X. Clearly, the union of any two closed sets is again a closed set.

[*For definition and results we refer to Simmons, G.F. (1) P. 95*]

Definition7: Open sphere Let (X, d) be a metric space. For, any point x_0 in X and any real number $r > 0$. Let

$$S(x_0, r= S_r(x_0) = \{x \in x: d(x_0, x) < r\}.$$

Then $S_r(x_0)$ is called an open sphere with center x_0 and radius r. An open sphere $S_r(x_0)$ is also called an r ball or simply open ball with Centre x_0 and radius r and then we denote it by $B_r(x_0)$ clearly an open sphere always contains its center because $d(x_0, x_0) = 0 < r$.

Definition 8:Neighborhood *A* subset V of a metric space X is called a neighborhood of a point $x \in X$ if V contains an open set G containing x.

Every open set G containing a point x is a neighborhood of x.

Definition 9: Interior point Let A be an arbitrary subset of a metric space X. A point $x \in X$ is called an interior point of A if there exists $r > 0$ such that $S_r(x) \subseteq A$.

That is if the point is the center of same open sphere $S_r(x)$ contained in A.

Definition 10: Interior The set of all interior points of A is called the interior of A and is denoted by Int. (A) or A^0. Thus,

$$\text{Int. } (A) = \{x: x \in A \text{ and } \exists\, r > 0 \text{ such that; } S_r(x) \subseteq A\}$$

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Result 5: Int *(A)* is an open set.

Result 6: *A* is open ⇔*A* = Int *(A)*

Definition 11: Accumulation point Let *A* be subset of a metric space *X*. A point *a* in *x* is called an accumulation point of *A* if every open sphere centered on a contains at least one point of *A* different from a. Accumulation point is also known as the limit point or cluster point.

Definition 12: Derived setthe set of all accumulation points of *A* is denoted by *A'* and is called the derived set of *A*.

Definition 13: Closure of A Let*A* be a subset of a metric space *X*, then the set *A*∪*A*' is denoted by $\overline{A}$and is called the closure of *A*.

Hence,$\overline{A}$=*A*∪*A*'

From which we can conclude that $A \subseteq \overline{A}$: $\overline{A} \subseteq \overline{A \cup B}$&$\overline{B} \subseteq \overline{A \cup B}$

Also, *A∪B*= *A∪B*

Definition 14: Convex set Let *S* be a non-empty subset of a linear space *E* now for *x, y* ∈ *S* and *λ, μ* ≥ *0*, then *S* is called a Convex Set whenever,

$$\lambda x + \mu y \in S \text{ for } \lambda + \mu = 1.$$

Definition 15: S-CONVEX SET: Let A be a set in a linear space *E*.

Now if for *x, y* ∈ *A* we have scalars □, □ □≥ *0*, □ + □ □≤ *1* such that □*x* + □*y* ∈ *A* then we shall say that set *A* is a S–Convex Set.

S-CONVEX HULL: Let A be a S–Convex Set in a linear space *E*. The intersection of all S–Convex Sets containing A will be known as S–Convex Hull and we shall denote it by *S(A)* and we shall read it as the S–Convex Hull of S-Convex Sets.

Main Results

In this section we establish some of the results using the definitions given in above section.

Theorem 1: Let A and B are two S-convex sets. Prove that $A \cap B$ is also an S-convex set.

Proof: Let A, B are S-convex sets. Now, we have to prove that $A \cap B$ is also an S-convex set.

Thus, for any $\alpha, \beta \geq 0, \alpha + \beta \leq 1$.

We have to prove that for $x, y \in A \cap B$, we have $\alpha x + \beta \in A \cap B$.

Since $x, y \in A \cap B$.

Thus $x, y \in A$ and A is an S-convex set. It gives $\alpha x + \beta y \in A$.

Similarly, for $x, y \in B$ and B is also an S-convex set. We have $\alpha x + \beta y \in B$.

Combining both, we get:

$$\alpha x + \beta \in A \cap B.$$

It gives us that $A \cap B$ is an S-convex set.

Theorem 2: The <u>union</u> of a sequence of S-convex sets is an S-convex set, if they form a <u>non-decreasing chain</u> for inclusion.

Proof: The proof of this is on similar lines as for the convex sets. So, we omit the details.

Theorem 3: Let *A* be an S – convex set in a topological vector space *X*. Then $\bar{A}$ (the closure of *A*) is also an S-Convex Set.

Proof: In order to prove the theorem we first of all prove that if □be any scalar and □ □⊆X (a topological vector space) then $\alpha\bar{\gamma} = \overline{\alpha\gamma}$.

For this, if □ = 0 then obviously $\alpha\bar{\gamma} = \alpha\bar{\gamma}$

However, if □ is nonzero that is if □ □≠*0* then we suppose that,*x* be any element of $\alpha\bar{\gamma}$then

x = □ $\bar{\varpi}$ where $\bar{\varpi} \in \bar{\gamma}$

Now, let *D* be a neighborhood of □ $\bar{\varpi}$ □

Hence,$\frac{1}{\alpha}D$ is a neighbourhood of $\bar{\varpi}$.

Since $\varpi \in \bar{\gamma}, \left(\frac{1}{\alpha}D\right) \cap \gamma \neq \varnothing$.

Thus, there exist y ∈□ □such that $\gamma = \frac{1}{\alpha}Z$ for some Z.

Therefore, *Z* = *xy*∈□ □ □□

Hence, *Z* ∈□ □ □ □ □⋂□*D.*

Hence,□ □ □ □ ⋂ *D* ≠∅.

It makes it clear that any neighborhood D of □ $\bar{\varpi}$ intersects □ □□

Hence, *x* = □ $\bar{\varpi} \in \overline{\alpha\gamma}$□

Hence, $x \in \overline{\alpha\gamma}$□

That is $x \in$□ □$\bar{\gamma} \Rightarrow x \in \overline{\alpha\gamma}$

Thus,$\alpha\bar{\gamma} \subseteq \overline{\alpha\gamma}$.…(5.1)

Again let ϖ□be any element of $\overline{\alpha\gamma}$then,

w =□$\left(\frac{1}{2}\varpi\right) = \alpha x$where$x = \frac{1}{2}\varpi$.

Let D be a neighborhood of *x* then □D is a neighborhood of $\bar{\varpi}$ □

Hence, $(\alpha D) \cap (\alpha\gamma) \neq \varphi$.

Thus, there exist an element $Z_1 \in$□ □D such that

Z_1 = □y Such that y ∈□D and y∈□□

Hence $y \in D \cap \gamma \Rightarrow D \cap \gamma \neq \varphi$.

But D is any neighborhood of *x*.

Hence any neighborhood of *x* intersects □.

This implies that $x \in$□$\bar{\gamma}$□

Hence $\overline{\omega}$ □ □ □ *x*□ ∈□$\bar{\gamma}$□

Hence $\varpi \in \overline{\alpha\gamma} \Rightarrow \varpi \in \alpha\bar{\gamma}$.

Thus $\overline{\alpha\gamma} \subseteq \alpha\bar{\gamma}$.…(5.2)

Hence from equations (5.1) and (5.2) we at once get,

$$\alpha\bar{\gamma} = \overline{\alpha\gamma.}$$

Now let □≥ 0, □ □≥□ □be such that □ □ □ □ □≤□ □□

Also let x, y be any two elements of $\bar{A}$.

Then $\alpha x + \beta y \in \alpha\bar{A} + \beta\bar{A} = \overline{\alpha A} + \overline{\beta A} = \bar{L} + \bar{T}$.

Now, let for a moment □A = L then $\overline{\alpha A} = \bar{L}$

□A = T. Then,$\overline{\beta A} = \bar{T}$.

We now take $a \in \bar{L}, b \in \bar{T}$.

Also let $\overline{\omega}$ be a neighborhood of a + b.

Also let there are neighborhoods W_1 and W_2 of a and b such that

$W_1 + W_2 \subset W$.

Then, there exist $x \in L \cap W_1$and $y \in T \cap W_2$.

Since, $a \in \bar{L}$and $b \in \bar{T}$then x + y must lie in (L+T) ∩W.

That is $x + y \in (L + T) \cap W$.

Therefore, $(L + T) \cap W \neq \varphi$.

Hence, $a + b \in \overline{L + T} = \overline{\alpha A + \beta B}$.

But by hypothesis *A* is a S-Convex Set in *X*.

Hence, □*A* + □*A* ⊆*A*.

Thus, $\overline{\alpha A + \beta B} \subseteq \bar{A}$.

Hence, □x + □y ⊆$\bar{A}$.

That is, we have that for $x, y \in \bar{A}$; α, βbe scalars and $\alpha + \beta \leq 1$.□x + □y is in $\bar{A}$.

Hence, $\bar{A}$is an S-Convex Set.

Theorem 4: Let

i) X be a topological vector space.

ii) A be an S–Convex Set in X.

iii) $\{0\}$ be an open set X.

Then, provethat A is closed.

Proof: By hypothesis A is an S-Convex Set.

Hence origin belongs to A (or equivalently A contains origin)

Hence, $0 \in A$

We also know that if $\bar{A} \subseteq A \Rightarrow A$is closed.

So, to prove the theorem, it is sufficient to show that $\bar{A} \subseteq A$

For this, since we already know that,

$$\bar{A} = \cap\, (A + V)$$

Where, V sums through all neighborhoods of Zero.

Since $\{0\}$ is also a neighborhood of 0.

Hence, $\bar{A} \subseteq A + \{0\} = A$

Thus, $\bar{A} \subseteq A$

Hence, A is closed.

That is A is a closed S–Convex Set.

Theorem 5: Let

i) X be a topological vector space.

ii) A be a S- convex set in X.

iii) $A^o \neq \varphi$.

Then, prove that A^o is also an S-Convex Set.

Proof: By definition, $A^o \subseteq A$.

Also, by hypothesis $A^o \neq \varphi$and $A^o$is open.

Hence, A^o cannot be a singleton set.

It means A^o contains at least two elements.

Let, $x, y \in A^o$andα, β be positive scalars such that $\alpha + \beta \leq 1$.

In case □ □= 0, □ = 1 or □ = 1, □ = 0

Clearly $\alpha x + \beta y \in A^o$

So let us assume that $\alpha <> 0, \beta > 0$ then,

$$\alpha x \in \alpha A^o \text{ and } \beta y \in \beta A^o.$$

Hence,$\alpha x + \beta y \in \alpha A^o + \beta A^o$.

Now, $\alpha A^o + \beta A^o = \underset{Ze\beta A}{U} (Z + \alpha A^o)$.

Since, $\alpha \neq 0, \alpha A^o$ is open and hence $Z + \alpha A^o$ is also open.

Hence, $\alpha A^o + \beta A^o$ being union of open sets is open.

Also $\alpha A^o + \beta A^o \subseteq \alpha A + \beta A \subseteq A$.

Hence by the definition of A^o,

$\alpha A^o + \beta A^o \subseteq A^o$.

Thus, $\alpha x + \beta y \notin > A^o$.

Hence, A^o is a S–convex set.

Theorem 6: Let

i) *X* be a topological vector space.

ii) *S* is an open set in *X*.

Then, the S–convex hull of *S* is also open.

Proof: Let *x* be in the S–convex hull of *S*.

Then, *x* is a finite sum $\Sigma\alpha_i x_i$ where $x_i \in S$ for i = 1, 2, 3, ..., n.

□$_i$=0 for each i = 1, 2, 3, …, n such that $\sum_{i=1}^{n} \alpha_i \leq 1$.

Since, S is open, then their exist neighborhoods V_i of x_i such that $V_i \subset S$.

Let *T* be the set $\Sigma\alpha_i v_i$.

That is, *T* is the set of all elements $\Sigma\alpha_i v_i$ where $v_i \in V_i$.

Hence, it is clear that $x \in T \subseteq$(the S-convex null of *S*).

Now to prove the theorem it is sufficient to prove that *T* is open.

For this, since each □$_i$ V_i is open, since $\alpha_i \neq 0$.

Hence by the method of induction *T* will be open.

As we already know that if *U* and *V* are open then *U* + *V* is open.

Now *x* + *V* is open for each fixed *x* and *U* + *V* is the union of all *x* + *V* as *x* varies over *U*.

Thus *U* + *V* is open.

Conclusion: In this article, we define the S-convex set and topological structure for the collection of these sets. We investigate fundamental ideas such as closed sets, open sets, set interiors, and the closure property of S-Convex Sets. In contrast to metric spaces, it proves that the closure of an S-Convex Set in a topological vector space stays S-Convex. Additionally, a conclusion about S-Convex Sets and S-Convex hulls is derived in this chapter. Our research paves the way to find the more interesting topological properties of new defined S-convex sets.

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