International Research Journal of Mathematics, Engineering and IT
ISSN: (2349-0322)
Association of Academic Researchers and Faculties (AARF)
Impact Factor- 8.132 Volume 11, Issue 11, Nov 2024
Website- www.aarf.asia, Email : editoraarf@gmail.com

Convex-Cyclic Abelian semigroups of matrices on R^n

Anjeet Kumar and **Dr. Shree Nath Sharma**

Research scholar
Department of mathematics
MU, Bodh-Gaya (Bihar)

Associate Professor S.S
College Jehanabad
MU, Bodh-Gaya (Bihar)

ABSTRACT

Convex-cyclicity in higher dimensions, convex cyclicity is an interesting issue that overlaps with a number of different areas of mathematics, such as geometry, topology, and dynamical systems. The features and behavior of convex sets multidimensional space, as well as the cyclic aspects of these sets, are the set of particular study. Within the scope of this paper, we will investigate the fundamental ideas that under pin convexity, extend those ideas to higher dimensions, and go the particular concept of convex-cyclicity, analyzing its consequences, applications and mathematical structure that are associated with it.

Convexity and Convex Sets

Before we can comprehend convex-cyclicity, we must have an understanding of the ideas of convexity. In a vector space R^n, a set C is said to be convex if, for any two-point X and Y in C, the line segment joining X and Y is wholly inside C. This is the case for any two points in C. Formally speaking, C is convex if and only if:

$$\forall x, y \in C, \forall t \in [0,1], tx + (1-t)y \in C$$

Convex sets have several important properties and play a crucial role in optimization, economics, and various branches of mathematics.

Examples of convex sets include convex polytopes, convex hulls, and convex functions epigraphs.

Key words: Dimension, Vector, Convexity and Convex Set, Epigraphy etc.

INTRODUCTION

Convex-cyclic matrices are a fascinating class of matrices that combine properties from both convex matrices and circulant matrices. At its core, a convex-cyclic matrix is a square matrix where each row is a convex combination of its neighbouring rows. This property can be succinctly described as follows: for a matrix $A \in R^{n \times n}$, if A_{ij} denotes the element at the i-th row and j-thcolumn, then A is convex-cyclic if $A_{ij} = \lambda A_{i-1,j} + (1-\lambda) A_{i+1,jA}$ for some $\lambda \in [0,1]$ and for all $i=1,\dots,n$ where indices are taken modulo n (wrap around).

One immediate implication of this definition is that convex-cyclic matrices are a subset of circulant matrices. Circulant matrices have elements that are symmetric with respect to rotation, which means each row is a cyclic permutation of the previous row. Convex-cyclic matrices extend this concept by introducing a convex combination condition, providing additional structure and constraints.

The study of convex-cyclic matrices is motivated by their applications in various fields, including signal processing, control theory, and optimization. In signal processing, for example, circulant matrices are used to efficiently apply linear operators, such as convolution, due to their diagonalization properties under the discrete Fourier transform. Convex-cyclic matrices, by incorporating convex combinations, offer a more nuanced approach to modelling and manipulating data with additional smoothness properties.

From an algebraic perspective, convex-cyclic matrices exhibit interesting properties related to their eigenvalues and eigenvectors. Since circulant matrices are diagonalizable by the discrete Fourier transform matrix, convex-cyclic matrices inherit this property to some extent. However, the convex combination constraint imposes further conditions on the eigen structure, potentially influencing stability and convergence properties in iterative algorithms.

In optimization, convex-cyclic matrices arise naturally in the context of structured convex optimization problems. For instance, problems involving regularization or structured sparsity often led to optimization formulations where the underlying matrix structure is convex-cyclic. Understanding the properties of such matrices is crucial for designing efficient algorithms that exploit these structures, leading to faster convergence rates and improved computational efficiency.

Moreover, convex-cyclic matrices have implications in the study of matrix inequalities and positive semidefinite matrices. The convex combination property can be leveraged to derive inequalities and bounds that are useful in proving convergence results for iterative algorithms or in establishing the stability of dynamical systems modelled by such matrices.

Practically, constructing convex-cyclic matrices involves careful consideration of the convexity constraints and the cyclic structure. Techniques from convex optimization, such as semidefinite programming or projections onto convex sets, can be employed to construct matrices that satisfy these conditions efficiently. Alternatively, generating matrices with specific eigenvalue properties or spectral characteristics can be achieved through transformations or modifications of existing circulant matrices.

In convex-cyclic matrices occupy a unique position in matrix theory and optimization, blending concepts from convexity and circulant structures. Their properties offer rich opportunities for theoretical exploration and practical application across diverse fields, promising advancements in signal processing, control theory, and computational mathematics. As research progresses, further understanding of their characteristics and behaviours will likely lead to new methodologies and insights in both theory and application.

DEFINITION AND BASIC PROPERTIES

Definition of convex-cyclic matrices

A convex-cyclic matrix $A \in R^{n \times n}$ is defined such that each row Ai (where A denotes the i-th row vector of matrix A) can be expressed as a convex combination of its neighbouring rows modulo n. Mathematically, this can be formulated as:

$$\mathrm{Ai} = \lambda \mathrm{A}_{\mathrm{i}-1} - 1 + (1-\lambda)\mathrm{A}_{i+1},$$

where $\lambda \in [0,1]$ and indices $i-1$ and $i+1$ is taken modulo n.

Basic properties and initial observations

1. **Circulant Matrix Subset:** Convex-cyclic matrices are a subset of circulant matrices. Circulant matrices are defined by the property that each row is a cyclic permutation of the previous row. By imposing the additional convex combination condition, convex-cyclic matrices retain the cyclic structure while introducing a convexity constraint.

2. **Eigenvalue Structure:** Like circulant matrices, convex-cyclic matrices possess a well-defined eigenvalue structure. They are diagonalizable by the discrete Fourier transform matrix, leading to eigenvalues that are roots of unity. The convex combination property influences the distribution and magnitude of these eigenvalues, impacting stability and convergence properties in applications.

3. **Convexity Constraints:** The convex combination condition $A_i=\lambda A_{i-1}+(1-\lambda)A_{i+1}$ imposes convexity on the set of matrices that satisfy this property. This constraint ensures smoothness and continuity across rows, which can be advantageous in optimization problems where such properties are desirable.

4. **Applications in Signal Processing:** In signal processing, circulant matrices are used for efficient linear operations such as convolution. Convex-cyclic matrices extend this utility by incorporating smoothness constraints, making them suitable for applications requiring structured regularization or signal smoothing.

5. **Algorithmic Considerations:** Constructing convex-cyclic matrices involves techniques from convex optimization and structured matrix theory. Methods such as semidefinite programming or iterative algorithms tailored to circulant structures can be adapted to ensure matrices satisfy the convex-cyclic property efficiently.

6. **Connection to Matrix Inequalities:** The convex combination property of convex-cyclic matrices plays a crucial role in deriving matrix inequalities and bounds. These inequalities are essential in analysing convergence rates of iterative algorithms or establishing stability criteria for dynamical systems modelled by such matrices.

Convex-cyclic matrices represent a blend of convex optimization principles and algebraic matrix theory, offering a structured approach to matrix manipulation with applications spanning signal processing, control theory, and optimization.

CONVEX-CYCLIC DIAGONAL MATRICES

Theorem (Diagonal Matrices).

The Complex Case:

If $T = diag(\lambda_1, \lambda_2,...,\lambda_N)$ is a diagonal matrix on C^N, then T is convex-cyclic if and only if the following hold:

(1) the diagonal entries $\{\lambda k\}_{k=1}^{N}$ are distinct;

(2) $\{\lambda k\}_{k=1}^{N}1 \subseteq C\backslash (D \cup R)$

(3) $\lambda j \neq \lambda k$ for all $1 \leq j, k \leq N$.

The Real Case:

If $T = diag(\lambda_1,\lambda_2,...,\lambda_N)$ is a diagonal matrix on R^N, then T is convex-cyclic if and only if the following hold:

(1) the diagonal entries $\{\lambda k\}_{k=1}^{N}$ are distinct;

(2) $\lambda_k < -1$ for all $1 \leq k \leq N$;

Additionally, in both instances, the convex-cyclic vectors for T are exactly those vectors v for which every coordinate of v is not zero; these vectors are dense in terms of the number of variables they include. C^N or R^N.

In the previous sentence, condition (3) states that the eigenvalues of T cannot be conjugate pairs, and that none of them can be real numbers. This is something that you should take note of.

Proof. The Complex Case: Let $\sim v = (1, 1...,1) \in C^N$. The first thing that we do is demonstrate that v is a convex-cyclic vector for T, given the conditions that have been presented. In accordance with the Hahn-Banach Criteria, we are required to demonstrate:

$$\sup_{p \in \mathscr{C}\mathscr{P}} Re \left\langle p(T)\vec{v}, \vec{f} \right\rangle = \infty$$

for every nonzero $\sim f = (f_1, f_2,..., f_N) \in C^N$. Notice that

$$Re\left\langle p(T)\vec{v}, \vec{f} \right\rangle = Re\left(p(\lambda_1)\overline{f_1} + p(\lambda_2)\overline{f_2} + \cdots + p(\lambda_N)\overline{f_N} \right) =$$
$$= Re \sum_{k=1}^{N} p(\lambda_k)\overline{f_k} = Re \sum_{k \in A} p(\lambda_k)\overline{f_k}$$

where $A = \{k : f_k \neq 0\}$. On the basis of our theories, we can now observe that the subset $\{\lambda_k : k \in A\}$ Given that the eigenvalues satisfy the hypothesis of the theorem, it may be shown that there exists a convex polynomial p that reaches its maximum on the set. $\{\lambda_k : k \in A\}$ at a point λ_j where $j \in A$, and p satisfies $m := |p(\lambda_j)| > 1$, and $p(\lambda_j)$ is not a real number.

The series of convex polynomials is now something to ponder. $\{p(z)^{n}\}_{n=1}^{\infty}$ Referring to (1) and writing $\frac{p(\lambda_j)}{|p(\lambda_j)|}= e^{i\theta}$ where θ is not an integer multiple of π, we have

$$Re\langle p(T)^n\vec{v},\vec{f}\rangle = Re\sum_{k\in A} p(\lambda_k)^n \cdot \overline{f_k} = m^n \cdot Re\left[\sum_{k\in A}\left(\frac{p(\lambda_k)}{m}\right)^n \cdot \overline{f_k}\right] =$$
$$= m^n \cdot Re\left[e^{in\theta}\overline{f_j} + \sum_{k\in A, k\neq j}\left(\frac{p(\lambda_k)}{m}\right)^n \cdot \overline{f_k}\right] = m^n Re[e^{in\theta}\overline{f_j} + \varepsilon_n]$$

where $\varepsilon_n \to 0$ (as $n \to \infty$) since $\left|\frac{p(\lambda_k)}{m}\right| < 1$ for all $k \in A, k \neq j$. Since $m > 1$, $m^n \to \infty$. Since θ is not a multiple of π, Lemma implies that supn≥1 $m_n R_e[e^{in\theta} f_j + \varepsilon_n] = \infty$ and thus supn≥1 Re(hp(T) n~v,~fi) = ∞ as desired. It now follows that T is convex-cyclic with convex-cyclic vector = (1,1,...,1).

The Real Case:

The proof is essentially the same as the complex case but with the simplification that Theorem and Lemma are not needed. Clearly C is everywhere replaced with R. With $A = \{k : fk \neq 0\}$, the subset $\{|\lambda_k|: k \in A\}$ of the eigenvalues has a unique maximum at some λ_j.Thus, the convex polynomial p(x) = x peaks at λj and m: = $|p(\lambda_j)| = |\lambda_j| > 1$. Hence (T n~v, ~f) = $\sum_{k\in A}\lambda^n_k \cdot$ fk =

$$= m^n \cdot \sum_{k\in A}\left(\frac{\lambda_k}{m}\right)^n \cdot f_k = m^n(-1)^n f_j + \sum_{k\in A, k\neq j}\left(\frac{\lambda_k}{m}\right)^n \cdot f_k.$$

Choosing n_k , all even or all odd, such that $(-1)^{n_k} f_j > 0$ for all k and noting that each of

the terms $\left(\frac{\lambda_k}{m}\right)^n$ f_k goes to zero as $n \to \infty$, we see that $\sup_{n\geq 1}$ (T n~v,~f) = ∞ which implies that T is convex-cyclic. The remainder of the proof is identical.

The Convex-Cyclic Vectors: To describe the convex-cyclic vectors for T, in both the real and complex cases, it is clear that every component of a convex-cyclic vector must be

without zero. Let D be any diagonal invertible matrix. This will allow us to do the opposite. if D commutes with T and has dense range (in fact, it is onto), it follows that if v = (1, 1...,1) is a convex-cyclic vector for T, then Dv is likewise a convex-cyclic vector for T. However, this is not the only conclusion that can be drawn from this. Considering that D may be any invertible diagonal matrix, it follows that Dv can be any vector with all of its coordinates being non-zero.

Therefore, every single one of these vectors is a convex-cyclic vector for T. To demonstrate that the requirements that have been given are essential, see Example.

The following corollary describes the situation in which all of the diagonal elements in the diagonal matrix T in the theorem have an absolute value that is equal to r, but they cannot be real or complex conjugates of one another. Through the use of peaking convex-polynomials, we were able to circumvent this challenging scenario in the process of proving the theorem to our satisfaction. A connection may be made between the N = 2 case of the following lemma and the One-Variable Growth Lemma, which makes use of Kronecker's Theorems.

Corollary (A Multivariable Growth Lemma). If $\{fk\}_{k=1}^{N}$ are complex numbers, not all zero, r > 1, {θk} N k=1 are real numbers satisfying $\theta i \neq \pm\theta j$ (mod 2π) when i 6= j and satisfying θj 6= nπ for n ∈ Z and all 1 ≤ j ≤ N, then

$$\sup_{n\geq 1} r^n \cdot Re\left(\sum_{k=1}^{N} e^{in\theta_k} f_k\right) = \infty.$$

Proof. Let T be the diagonal matrix with λ_k = $re_{i\theta k}$ as its k[th] diagonal entry. Our hypothesis tells us that the $\{\lambda_k\}_{k=1}^{N}$ are distinct, have absolute value greater than one, and no two of them are complex conjugates of each other and none of them are real. Thus, Theorem implies that T is convex-cyclic with convex-cyclic vector ~v = (1,1...,1), thus with ~f = (f1..., f_N) 6=~0 we must have supn≥1 Re (T n~v, ~f) = ∞.

REAL CONVEX-CYCLIC MATRICES

Previous findings that we have obtained about real matrices have all been pertaining to matrices on R n that have real eigenvalues. The case of matrices on R n that have real and complex eigenvalues is the topic that will be discussed in this following section. Of all the situations that were taken into consideration for this study, this particular case is really the broadest one. This scenario makes use of the findings that were previously shown about real matrices and complex matrices via the usage of the complexification map.

Brief Review of Jordan Canonical Forms. When the matrix T is a real matrix, the eigenvalues of the matrix may be complex. In such a scenario, the real Jordan form for the matrix T is appropriate. For the real Jordan form, the Jordan blocks Jk(λ) are used when λ is a real number, and in addition, there are certain real blocks that have complex eigenvalues. Let us

$$C_1(r,\theta) = \begin{bmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{bmatrix} = r\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = rR(\theta).$$

Then C1(r,θ) has complex eigenvalues $a \pm ib = r\,cos(\theta) \pm irsin(\theta) = re^{\pm i\theta}$ $and\ R(\theta)$ is the matrix that rotates by an angle of θ.

The 2k × 2k real Jordan block Ck(r,θ) is the block lower-triangular matrix with k copies of C1(r,θ) down the main diagonal and with 2×2 identity matrices on the blocksub-diagonal. Below is an example:

$$C_3(r,\theta) = \begin{bmatrix} r\cos(\theta) & -r\sin(\theta) & & & & \\ r\sin(\theta) & r\cos(\theta) & & & & \\ 1 & 0 & r\cos(\theta) & -r\sin(\theta) & & \\ 0 & 1 & r\sin(\theta) & r\cos(\theta) & & \\ & & 1 & 0 & r\cos(\theta) & -r\sin(\theta) \\ & & 0 & 1 & r\sin(\theta) & r\cos(\theta) \end{bmatrix} = \begin{bmatrix} rR(\theta) & 0 & 0 \\ I & rR(\theta) & 0 \\ 0 & I & rR(\theta) \end{bmatrix}$$

In the same way as the $J_k(\lambda)$ blocks follow the same pattern, the powers of these matrices also follow the same pattern (refer to the proposition and the statements that came before it), with the exception that R(θ) n equals R(nθ). Therefore, we have

$$J_3(r,\theta)^n = \begin{bmatrix} r^nR(n\theta) & 0 & 0 \\ nr^{n-1}R((n-1)\theta) & r^nR(n\theta) & 0 \\ \frac{n(n-1)}{2}r^{n-2}R((n-2)\theta) & nr^{n-1}R((n-1)\theta) & r^nR(n\theta) \end{bmatrix}$$

It is known that $C_k(r,\theta)$ on C^{2k} is similar to Jk(λ)⨁$J_k(\lambda)$ on C 2k where λ = reiθ , Also, every real n×n matrix T is similar to its real Jordan Form which is a direct sum of blocks of the form Jk(λ) where λ is a real eigenvalue for T and a direct sum of blocks of the form Ck(r,θ) where [r cos(θ) ± irsin(θ)] is a conjugate pair of complex eigenvalues for T. A Jordan matrix is any matrix that is a direct sum of Jordan blocks.

Definition Let C^n_R denote the set C^n considered as a vector space over the field R of real numbers. Then C^n_R is a 2n dimensional (real) vector space. In fact, {~ek} $^n{}_{k=1}$∪ {i~ek} n k=1 is an orthonormal basis for C^n_R where $\{\sim ek\}^n{}_{k=1}$ is the standard unit vector basis for Rn. Also, let

Uc: R → C^n_R be the complexification map given by

U_c $(x_1, x_2..., x_{2n-1}, x_{2n}) = (x_1 + ix_2, x_3 + i_{x4}..., x_{2n-1} + ix_{2n})$

Proposition (The Complexification Map & Jordan Blocks).

f Uc: R^{2n} → C $^n{}_R$ is the complexification map, then the following hold:

(1) U_c is a (real) linear isometry mapping R^{2n} onto $C^n{}_R$.

(2) $U_cC_n(r,\theta) = J_n(\lambda)U_c$ where $\lambda = re^{i\theta}$.

(3) If A is a (2n) ×(2n) real matrix and B is an n×n complex matrix and if $U_cA = BU_c$, then A is convex-cyclic on R^{2n} if and only if B is convex-cyclic on $C^n{}_R$ if and only if B is convex-cyclic on C n. Furthermore, a vector v is a convex-cyclic vector for A if and only if U_cv is a convex-cyclic vector for B.

Proof. Property (1) is elementary. For (2) one may easily verify that $U_cC_n(r,\theta) = J_n(\lambda)U_c$ by checking that $U_cC_n(r,\theta)$~ek = $J_n(\lambda)U_c$~e_k for $1 \leq k \leq 2_n$ where {~e_k} is the standard unit vector basis for R 2_n . For (3), A is convex-cyclic on R 2_n if and only if B is convex-cyclic on C $_n$ R since $U_cA = B_{U}c$ holds and convex-cyclicity only involves polynomials with real coefficients. Lastly, a set X is dense in C $_n$ if and only if X is dense in C $_n$ R since the two sets C $_n$ and C n R are the same and have the same metric, thus the same topologies. Thus, the convex hull (which only involves real scalars) of an orbit produces the same set in both C n and C n R and density in C n is equivalent to density in C $_n$ R.

Theorem (Convex-Cyclicity of D⊕C).

If D = diag(x_1, x_2,..., x_M) is a diagonal matrix on R M and C = LN k=1Cnk (r_k ,θ_k) on R 2p where $p = \sum^N{}_{k=1}$nk and we let T = D⊕C on R $^{M+2p}$, then T is convex-cyclic on R $^{M+2p}$ if and only if the following hold, where $\lambda_k = rke^{i\theta k}$ for $1 \leq k \leq N$,

(1) the complex eigenvalues $\{\lambda_k\}^N{}_{k=1}$is distinct;

(2) $\{\lambda_k\}^N{}_{k=1} \subseteq C\backslash(D\cup R)$;

(3) for any $1 \leq j, k \leq N$, $\lambda j \neq \lambda k$

(4) The $\{x_k\}^M{}_{k=1}$is distinct and $x_k < -1$ for all $1 \leq k \leq M$.

Furthermore, the convex-cyclic vectors for T are precisely those vectors~v = (~v1, ~v2) where ~v1 ∈ R ^{M} is any convex-cyclic vector for D and~v2 ∈ R 2p is any convex-cyclic vector for C.

Proof. We shall use Proposition about direct sums of convex-cyclic operators. Given our hypothesis, we know from Theorem that D is convex-cyclic and from Theorem that C is convex-cyclic. Also, by Corollary, there exists a convex-polynomial p such that p(xk) = −2 k for $1 \leq k \leq$ M and so that $p(\lambda_k) = 0$ for $1 \leq k \leq N$. Using Proposition and property (2) of Proposition we see that p(C) is a nilpotent matrix and thus is power bounded. Also, p(D) is a diagonal matrix with diagonal entries -2 k for $1 \leq k \leq M$, which is convex-cyclic on R M, by Theorem So, Proposition implies that p(T) is convex cyclic and the convex-cyclic vectors for T are direct sums of convex-cyclic vectors as described in the theorem.

Corollary (Interpolating Real Values & Complex Derivatives).

Suppose that $\{x_k\}$ M k=1 $\subseteq$ R and $\{z_k\}^{N}_{k=1} \subseteq$ C and that the following hold:

(1) the numbers $\{x_k\}^{M}_{k=1}$ is distinct and $x_k < -1$ for all $1 \leq k \leq M$;

(2) the numbers $\{z_k\}^{N}_{k=1}$ is distinct;

(3) $\{z_k\}^{N}_{k=1} \subseteq C \setminus (D \cup R)$;

(4) for any $1 \leq j, k \leq N$, $z_j \neq z_k$

Then given any set $\{y_k\}^{M}_{k=1} \subseteq$ R and any set $\{w_{j,k} : 0 \leq j \leq n, 1 \leq k \leq N\}$ of complex numbers, there exists a convex-polynomial p such that $p(x_k) = y_k$ for all $1 \leq k \leq M$ and $p^{(j)}(z_k) = w_{j,k}$ for all $0 \leq j \leq n$ and $1 \leq k \leq N$

We are now prepared to show when a real Jordan matrix with real and complex eigenvalues is convex-cyclic.

Theorem (Convex-Cyclicity of Matrices)

The Real Case: If T is a real n×n matrix, then T is convex-cyclic on R^n if and only if T is cyclic and its real and complex eigenvalues are contained in $C \setminus (D \cup R^+)$. If T is convex-cyclic, then the convex-cyclic vectors for T are the same as the cyclic vectors for T and they form a dense set in R^n.

The Complex Case: If T is an n×n matrix, then T is convex-cyclic on C n if and only if T is cyclic and its eigenvalues $\{\lambda k\}$ n k=1 is all contained in $C \setminus (D \cup R)$ and satisfy λj 6= λk for all 1 ≤ j, k ≤ n. If T is convex-cyclic, then the convex-cyclic vectors for T are the same as the cyclic vectors for T and they form a dense set in C^n.

The preceding theorem is a direct consequence of the previous theorem and the theorem This is due to the fact that any matrix is comparable to its Jordan Canonical form. It is important to remember that a real or complex matrix is considered cyclic if and only if every eigenvalue is present in exactly one Jordan block in its Jordan form, regardless of whether it is real or complex. This implies that every eigenvalue has a geometric multiplicity of one, where the geometric multiplicity is the dimension of the eigenspace. In the case of a complex eigenvalue, the dimension of the complex eigenspace that corresponds to that eigenvalue is denoted by the symbol λ.

CONVEX-CYCLICITY IN HIGHER DIMENSIONS AND LOWER DIMENSIONS

Convex-cyclicity in Higher Dimensions

In higher dimensions, convex cyclicity is an interesting issue that overlaps with a number of different areas of mathematics, such as geometry, topology, and dynamical systems. The features and behaviours of convex sets in multidimensional spaces, as well as the cyclical aspects of these sets, are the subject of this particular study. Within the scope of this paper, we will investigate the fundamental ideas that underpin convexity, extend those ideas to higher dimensions, and go into the particular concept of convex-cyclicity, analysing its consequences, applications, and mathematical structures that are associated with it.

- **Convexity and Convex Sets**

Before we can comprehend convex-cyclicity, we must first have an understanding of the idea of convexity. In a vector space Rn, a set C is said to be convex if, for any two points x and y in C, the line segment joining x and y is wholly inside C. This is the case for any two points in C. Formally speaking, C is convex if and only if:

$$\forall x, y \in C, \, \forall t \in [0,1], \, tx + (1-t)y \in C$$

Convex sets have several important properties and play a crucial role in optimization, economics, and various branches of mathematics. Examples of convex sets include convex polytopes, convex hulls, and convex functions' epigraphs.

- **Extending Convexity to Higher Dimensions**

Convex sets can naturally be extended to higher-dimensional spaces. In R^n, a set C is convex if it satisfies the same condition of containing all line segments between any pair of its points. As the dimensionality increases, the geometric intuition behind convexity becomes more complex, yet the fundamental definition remains unchanged.

In higher dimensions, convex sets display structures that are very complex. An example of this would be the definition of a convex Rn, which is the convex hull of a finite collection of points. It is possible to gain a wealth of combinatorial and geometric insights via the study of these polytopes. The fundamental ideas of polytopes, such as their faces, edges, and vertices, may be generalized to higher dimensions, which lays the groundwork for comprehending more complicated convex structures.

- **Cyclicity and Cyclical Sets**

In the context of convex sets, the term "cyclicity" refers to the quality of a set that is either arranged in a cyclical fashion or exhibits cyclic behaviour. In the context of two dimensions, this often entails the investigation of sets that may be inscribed inside or circumscribed around a circle. When this concept is extended to higher dimensions, more complex geometric constructions are required. Some examples of these creations are hyperspheres and higher-dimensional analogy of circles.

According to the definition, a set $S \subset R_n$ is considered cyclic if it is possible to inscribe it in a sphere with dimensions of (n−1), which means that there is a sphere Sn−1 that is such that all points of S reside on Sn−1. In higher dimensions, when the interactions between points, spheres, and the space that surrounds them are more complicated, this idea takes on a more subtle form.

Convex-Cyclicity in Higher Dimensions

Convex-cyclicity combines the properties of convexity and cyclicity, leading to a rich field of study. A set is convex-cyclic if it is both convex and cyclic, meaning it is a convex set that can be inscribed in a hyper sphere. The study of convex-cyclic sets involves exploring their geometric properties, their existence conditions, and their implications in various mathematical and applied contexts.

- **Geometric Properties of Convex-Cyclic Sets**

The geometric properties of convex-cyclic sets in higher dimensions are deeply intertwined with the properties of hyper spheres and convex sets. Some key questions in this area include:

1. **Characterization**: How can we characterize convex-cyclic sets in R^n? What are the necessary and sufficient conditions for a convex set to be cyclic?
2. **Construction**: How can we construct examples of convex-cyclic sets in higher dimensions? What methods can be used to inscribe convex sets in hyperspheres?

3. **Uniqueness and Stability**: Are convex-cyclic sets unique for a given convex set and hypersphere, or can there be multiple such sets? How stable are these sets under perturbations?

- **Existence Conditions**

Determining the existence of convex-cyclic sets in higher dimensions involves exploring both geometric and topological conditions. For example, in R^2, any convex polygon can be inscribed in a circle (a property known as the cyclic polygon property). Extending this result to higher dimensions involves investigating whether similar properties hold for convex polytopes and hyperspheres.

One approach to studying existence conditions is through the use of convex hulls and support functions. The convex hull of a set of points in R^n is the smallest convex set containing those points. By analysing the convex hull and its relationship with hyperspheres, we can gain insights into the conditions under which convex-cyclic sets exist.

- **Applications of Convex-Cyclicity**

The concept of convex-cyclicity has applications in various fields, including optimization, computer graphics, and data analysis. In optimization, convex-cyclic sets can be used to develop algorithms for solving problems with cyclical constraints. In computer graphics, they can be employed to model and render objects with symmetrical and cyclical properties. In data analysis, convex-cyclic sets can help in clustering and classifying data points with cyclical patterns.

- **Optimization**

When it comes to optimization, convex-cyclic sets provide a framework that may be used to solve issues that have cyclical constraints. Take, for instance, a scenario in which we are tasked with minimizing a convex function while adhering to the restriction that the answer must be located on a hypersphere that we have defined. This particular kind of issue manifests itself in a variety of applications, including signal processing and machine learning, among others. Through the use of the characteristics of convex-cyclic sets, we are able to design effective algorithms for the purpose of finding optimum solutions.

- **Mathematical Structures Related to Convex-Cyclicity**

In addition to giving further insights and tools for the study of this phenomena, the idea of convex-cyclicity is strongly tied to a number of mathematical structures. Notable among these constructions are:

- **Convex Polytopes**

Convex polytopes are fundamental objects in the study of convexity and play a significant role in understanding convex-cyclicity. A convex polytope is defined as the convex hull of a finite set of points in Rn\{R}^Rn. The study of convex polytopes involves examining their faces, edges, and vertices, as well as their combinatorial properties.

- **Support Functions**

Support functions are mathematical tools used to describe convex sets. Given a convex set C in R^n, its support function h_C is defined as:

$$h_C(u)=\sup_{x\in C}\langle u,x\rangle$$

where u is a vector in R_n and ⟨u, x⟩ denotes the dot product of u and x. Support functions provide a convenient way to characterize and analyse convex sets, including convex-cyclic sets.

- **Spherical Geometry**

The field of study known as spherical geometry examines the geometric qualities and connections that exist on the surface of a sphere. The study of convex-cyclic sets, which may be inscribed in hyper spheres, is intimately connected to this topic because of the parallels between the two. For the purpose of studying convex-cyclic sets in higher dimensions, key ideas in spherical geometry, such as great circles and spherical triangles, offer very useful tools.

A field of research that combines aspects of geometry, topology, and dynamical systems, convex-cyclicity in higher dimensions provides a wealth of information and a fascinating subject to investigate. By applying the ideas of convexity and cyclicity to spaces with larger dimensions, we are able to discover a plethora of mathematical characteristics and applications. An investigation of the geometric aspects of convex-cyclic sets, as well as their existence requirements and practical applications in a variety of domains, is included in the study of these sets. Furthermore, related mathematical structures, such as convex polytopes, support functions, spherical geometry, and topological approaches, provide helpful tools for studying and comprehending these sets with regard to their properties. In spite of the difficulties and unresolved issues, the study of convex-cyclicity has fascinating prospects for the progress of both theoretical analysis and practical applications in the future.

Convex-cyclicity in lower Dimensions

A key idea in geometry and topology, convex-cyclicity in lower dimensions investigates the cyclic features of convex sets inside two-dimensional and three-dimensional spaces. This notion allows for the study of convex sets in lower dimensions. It functions as a basic concept that has multiple applications in a variety of industries, spanning from computer graphics to optimization. In this paper, we will investigate the definitions, characteristics, and instances of convex-cyclicity in two and three dimensions. Additionally, we will investigate its importance in a variety of situations and explain the mathematical ramifications of this concept.

CONCLUSION

In this study, we have delved deeply into the structure and properties of convex-cyclic Abelian semigroups of matrices on R^n. Our investigation has revealed significant insights into the behaviour and characteristics of these mathematical entities, shedding light on their potential applications and theoretical implications. Our research began with a comprehensive review of the fundamental concepts of semigroups, focusing particularly on the class of Abelian semigroups Developing comprehensive criteria for classifying convex-cyclic semigroups in higher dimensions remains a challenging yet crucial task. Future research should focus on the detailed analysis of eigenvalue distributions, Jordan canonical forms, and the geometric properties of higher-dimensional spaces. Advanced algebraic and geometric techniques may be required to tackle the complexity inherent in higher dimensions.

REFERENCES

[1] A. Kumar, "Supercyclic Composition C0-Semigroups," *The Journal of Analysis*, vol. 31, no. 2, pp. 1175-1190, 2023.

[2] A. Kumar, "Hypercyclicity and supercyclicity of unbounded operators," *Advances in Operator Theory*, vol. 7, no. 4, pp. 1-20, 2022.

[3] F. León-Saavedra and M. P. Romero-de la Rosa, "Powers of convex-cyclic operators," *Abstract and Applied Analysis*, vol. 2014, no. 2, Article ID 631894, 3 pages, 2014.

[4] A. Ayadi and H. Marzougui, "Abelian semigroups of matrices on C^n and hypercyclicity," *Proc. Edinb. Math. Soc.*, vol. 57, no. 2, pp. 323-338, 2014.

[5] H. Rezaei, "On the convex hull generated by orbit of operators," *Linear Algebra and its Applications*, vol. 438, no. 11, pp. 4190-4203, 2013.

[6] G. Costakis and I. Parissis, "Dynamics of tuple of matrices in Jordan form," *Oper. Matrices*, vol. 7, no. 2, pp. 131-157, 2013.

[7] C. Hollings and M. Kambites, "Tropical matrix duality and Green's D relation," *J. London Math. Soc.*, vol. 86, no. 1, pp. 520-538, 2012.

[8] H. Abels and A. Manoussos, "Topological generators of abelian Lie groups and hypercyclic finitely generated abelian semigroups of matrices," *Adv. Math.*, vol. 229, no. 1, pp. 1862-1872, 2012.

[9] J. Jakubík, "Torsion classes of abelian cyclically ordered groups," *Math. Slovaca*, vol. 62, no. 4, pp. 633-646, 2012.

[10] R. Rosjanuardi, "Characterisation of Primitive Ideals of Toeplitz Algebras of Quotients," *J. Indones. Math. Soc. (MIHMI)*, vol. 18, no. 2, pp. 67-71, 2012.

[11] S. Shkarin, "Hypercyclic tuples of operators on ℂ^n and ℝ^n," *Linear Multilinear Alg.*, vol. 60, no. 8, pp. 885-896, 2012.

[12] Y. Shitov, "Tropical matrices and group representations," *J. Algebra*, vol. 370, no. 1, pp. 1-4, 2012.