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SOME EXACT STATIC SPHERICALLY SYMMETRIC SOLUTIONS OF EINSTEIN'S FIELD EQUATIONS FOR THE ZELDOVICH FLUID DISTRIBUTION

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ABSTRACT :

This paper provides some exact, static spherically symmetric solution of Einstein's field equations for the Zeldovich fluid distributions in different cases. Various physical and geometrical properties have been studied.

Key Words : exact solution, Zeldovich fluid, perfect fluid, pressure, density.

1. Introduction

Various researchers in theory of relativity have focused their mind to the study of solution of Einstein's field equation using equation of state $p = \rho$ e.g., Latelier [11], Letelier and Tabensky [12] and Yadav et. al. [27]. Singh and Yadav [17] have also discussed the static fluid sphere with the equation of the state $p = \rho$. Further study in the line has been done by Yadav and Saini [25], which is more general than on due to Singh and Yadav [17]. Also in this case the relative mass m of a particle in the gravitational field is related to its proper mass m_0 as studied by Narlikar [13]. Schwarzschild [15] considered the perfect fluid spheres with homogeneous density and isotropic pressure in general relativity and obtained the solutions of relativistic field equations. Tolman [21] developed a mathematical method for solving Einstein's field equations applied to static fluid spheres in such a manner as to provide explicit solutions in terms of known analytic functions. A number of new solutions were thus obtained and the properties of three of them were examined in detail. Solution to Einstein's equations with a simple equations of state have been found in various cases, e.g. fo $\rho + 3p =$ constant (Whittaker [24]), for $\rho = 3p$ (Klein Singh and Abdussatar [16], Fenstein and

Senovilla [7])'; for p = ρ + constant (Buchdahl and Land [4], Allunutt [1]) and for $\rho = (1+a)\sqrt{p} + ap$ (Buchdahl [2]). But if one takes, e.g. polytrophic fluid sphere $\rho = ap^{1+\frac{1}{p}}$ (Klein [9] Tooper [22], Buchdahl [3]) or a mixture of ideal gas (Suhonen [18]) one soon has to use numerical methods. Davidson [5] has presented a solution for a non stationary analogue to the case when $p = \frac{1}{3}\rho.$ Tolman [21], Thomas E Kiess [20], Karmer [10], Singh et. al. [16], Raychaudhari [14], Walecka [23], Yadav, et. al. [28-29] and Yadav and Singh [26] are some of the authors who have studied various aspects of interacting fields in the framework of Einstein's field equations for the perfect fluid with specified equation of state in general relativity.

In this paper we have obtained some exact static spherically symmetric solution of Einstein field equation for the Zeldovich fluid distribution using equation of state p = ρ. It has been obtained taking suitable choice of g_{11} and g_{44} (e.g. $e^{\beta} = Ar^{\frac{\mu n^2+\upsilon n+3}{2n+1}}$, Where A is a constants). For different values of n and suitable choice of constants we get many previously known solutions. To overcome the difficulty of infinite density at the centre, it is assumed that distribution has a core of radius r_0 and constant density ρ_0 which is surrounded by the fluid with the Zeldovich fluid (i.e. p = e). Various physical and geometrical properties have been also found and discussed.

2. The Field Equations

We use the static spherically symmetric metric given by

$$ds^2 = e^{\beta}dt^2 - e^{\alpha}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.1}$$

where α and β are function of r only. The field equations

$$R^i_j - \frac{1}{2}R\delta^i_j = -8\pi T^i_j \tag{2.2}$$

for the metric (2.1) for the Zeldovich fluid which can be regarded as a perfect fluid having the energy momentum tensor

$$T^i_j = (\rho + p)u^i u_j - \delta^i_j p \tag{2.3}$$

Characterized by the equation of state p = ρ in comoving co-ordinates (i.e. $u_1 = u_2 = u_3 = 0$ and $u_4 = e^{-\frac{\beta}{2}}$) are

(2.4) $8\pi p = e^{-\alpha}\left(\frac{\beta'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}$

(2.5) $8\pi p = e^{-\alpha}\left(\frac{\beta''}{r}-\frac{\alpha'\beta'}{4}+\frac{\beta'^2}{4}+\frac{\beta^1-\alpha^1}{2r}\right)$

(2.6) $8\pi\rho = e^{-\alpha}\left(\frac{\alpha'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}$

3. Solutions of the Field Equations

From equations (2.4), (2.6) and using p = ρ we have

(3.1) $e^{-\alpha}\left(\frac{\beta'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}=e^{-\alpha}\left(\frac{\alpha'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}$

Equation (3.1) shows that if β is known α can be found, so we choose

Case I

(3.2) $e^{\beta} = Ar^{\left(\frac{\mu n^2+\upsilon n+3}{2n+1}\right)}$

where A is constant :

To avoid mathematical complexcity, we choose μ = 0, $\upsilon = 0$ and n = 1

so that (3.2) reduces to

(3.3) $e^{\beta} = Ar$

Using (3.3) equation (3.1) takes the form

(3.4) $\frac{e^{-\alpha}\alpha'}{r}-\frac{3e^{-\alpha}}{r^2}+\frac{2}{r^2}=0$

Putting z = $e^{-\alpha}$ the equation (2.9) is reduced to

(3.5) $\frac{dz}{dr}+\frac{3z}{r}=\frac{2}{r}$

which is a linear differential equation whose solution is

(3.6) $z=\frac{1}{r}+\frac{c}{r^3}$

Therefore we get

(3.7) $\frac{1}{r}+\frac{c}{r^3}$

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where C is an integration constant.

If we set c = 0, then

(3.8) $$e^{-\alpha} = \frac{1}{r}$$

Hence the metric (2.1) after suitable adjustment of constants takes the form

(3.9) $$ds^2 = r\,dt^2 - r^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

For the metric (3.9) the fluid velocity u^i is found to be

(3.10) $u^1 = u^2 = u^3 = U_1 = U_2 = U_3 = 0$ and $U^4 = \frac{1}{\sqrt{r}}, u_4 = \sqrt{r}$

The scalar of expansion $\theta = u^i; i$ is identically zero.

The non-zero components of the tensor of rotation w_{ij} and shear tensor σ_{ij} are

(3.11) $$W_{14} = -W_{41} = \frac{1}{2\sqrt{r}},$$

(3.12) $$\sigma_{14} = \sigma_{41} = -\frac{1}{2\sqrt{r}}$$

Case II : Here we choose $\mu = 1$, $\upsilon = 0$, n = 2 in (3.2)

we get

(3.13) $$e^B = Ar^{7/5}$$

Now use of (3.13) in (3.1) gives

(3.14) $$\frac{de^{-\alpha}}{dr} + \frac{17}{5r}e^{-\alpha} = \frac{2}{r}$$

Putting $Z = e^{-\alpha}$, (3.14) goes to the form

(3.15) $$\frac{dZ}{dr} + \frac{17}{5r}Z = \frac{2}{r}$$

which is a linear differential equation whose solution is given by

(3.16) $$Z = \frac{k}{r^{17/5}} + \frac{10}{17}$$

(3.17) $$e^{-\alpha} = \frac{K}{r^{17/5}} + \frac{10}{17}$$

where k is constant of integration

Hence metric (2.1) yields

(3.18) $ds^2 = Ar^{7/5}dt^2 - \left(\frac{k}{r^{17/5}} + \frac{10}{17}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

Absorbing the constant A in the coordinate differential dt and taking K = 0 (to avoid singularity). The metric (3.18) takes the form

(3.19) $ds^2 = r^{7/5}dt^2 - \frac{17}{10}dr^2 - r^2(d\theta^2 + \sin^2\theta + \sin^2\theta . d\varphi^2)$

The non-zero component of Riemann-christoffel curvature tensor R_{hijk} for the metric (3.19) is

(3.20) $\sin^2\theta R_{2424} = R_{3434} = \frac{17}{10} r^{7/5} \sin^2\theta = R_{2323}$

For the metric (3.19) the fluid velocity v^1 is given by

(3.21) $\nu^1 = \nu^2 = \nu^3 = 0, \quad \nu^4 = r^{-7/10} = \frac{1}{r^{7/10}}$

The scalar of expansion $e = \nu_i^j$ is identically zero (i.e., $e = 0$). The non vanishing components of the tensor of rotation ω_{ij} is defined by

$-\omega_{ij} = \nu_{ij} - \nu_{ji}$

are given by

(3.22) $\omega_{14} = -\omega_{41} = \frac{-7}{10} r^{-3/10} = \frac{-7}{10r^{3/10}}$

The components of the shear tensor σ_{ij} defined by

$\sigma_{ij} = \frac{1}{2}\left(\nu_{ij} + \nu^{ij}\right) - \frac{1}{3}H_{ij}$

where projection tensor

$H_{ij} = g_{ij} - \nu_i \nu_j$

are given by

(3.21) $\sigma_{14} = \sigma_{41} = \frac{7}{10} r^{-3/10} = \frac{7}{10r^{3/10}}$

with other components being zero.

Case III :

Here we choose $\mu = 0, \upsilon = \frac{1}{3}, n = \frac{3}{2}$ in (3.2) we get

(3.22) $e^{\beta} = Ar^{5/4}$

where A is constant

using (3.22) in (3.1), we obtain

(3.23) $$\frac{de^{-\alpha}}{dr} + \frac{13}{4r}e^{-\alpha} = \frac{2}{r}$$

Putting Z = $e^{-\alpha}$, the equation (3.23) goes to the form

(3.24) $$\frac{dz}{dr} + \frac{13}{4r}z = \frac{2}{r}$$

Which is a L.D.E. whose solution is

(3.25) $$z = \frac{8}{13} + \frac{k}{r^{13/4}}$$

or

(3.26) $$e^{-\alpha} = \frac{k}{r^{13/4}} + \frac{8}{13}$$

where k is constant of integration

Hence the metric (3.1) provides

(3.27) $$Ar^{5/4}dt^2 - \left(\frac{k}{r^{13/4}} + \frac{8}{13}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Absorbing the constant A in co-ordinate differential dt and putting K = 0, the metric (3.27) takes the form

(3.28) $$ds^2 = r^{5/4}dt^2 - \frac{13}{8}dr^2 - r^2(d\theta^2 + \sin^2\theta + d\phi^2)$$

The non-zero component of Relmann-christoffel curvature tensor R_{hijk} for the metric (3.28) is

(3.29) $$\sin^2\theta R_{2424} = R_{3434} = \frac{13}{8}r^{13/4}\sin^2\theta = R_{2323}$$

For the metric (3.28) the fluid velocity v' is given by

(3.30) $$v^1 = v^2 = v^3 = 0, \quad v^4 = r^{-\frac{5}{8}} = \frac{1}{r^{5/8}}$$

In the usual notation, we have the scalar of expansion, rotation and shear tensor as follows for metric (3.28) :

(3.31) $e = 0, \quad \omega_{14} = -\omega_{41} = \frac{-5}{8} r^{-3/8} = \frac{-5}{8r^{3/8}}$

and

(3.32) $\sigma^{14} = \sigma_{41} = \frac{5}{8} r^{-3/8} = \frac{5}{8r^{3/8}}$

From (3.1) we see that if α is known, then β can be obtained, so we choose.

(3.33) $e^{\alpha} = \lambda$

where λ is constant.

using (3.33), equation (3.1) yields

(3.34) $\beta' - \alpha' + \frac{2}{r}(1-\lambda) = 0$

Also since $e^{\alpha} = \lambda(\text{constant}) \Rightarrow \alpha' = 0$ and there fore

from (3.34) we have

(3.35) $\beta' + \frac{2}{r}(1-\lambda) = 0$

Which after integration yields

(3.36) $e^{\beta} = \bar{\mu} r^{2(\lambda-1)}$

where $\bar{\mu}$ is constant of integration.

Now (2.4) and (2.5) lead to $\lambda = 2$ so that

(3.37) $e^{\beta} = \bar{\mu} r^2$

Hence metric (2.1) may be written as

(3.38) $ds^2 = -2dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) + \bar{\mu} r^2 dt^2$

This metric can be put in the following form (by absorbing constant $\bar{\mu}$ in coordinate differential dt)

(3.39) $ds^2 = -2\, dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) + r^2 dt^2$

The non-zero components of the Riemann Christoffel curvature tensor R_{hijk} for the line element (3.39) are

(3.40) $R_{2424}\sin^2\theta = -R_{2323} = R_{3434} = \frac{1}{2}r^2\sin^2\theta$

Choosing orthonormal tetrad $\bar{\lambda}_{(j)^i}$ as,

(3.41) $$\begin{cases} \bar{\lambda}_{(1)^i} = \left(\frac{1}{\sqrt{2}}, 0, 0, 0\right) \\ \bar{\lambda}_{(2)^i} = \left(0, \frac{1}{r}, 0, 0\right) \\ \bar{\lambda}_{(3)^i} = \left(0, 0, \frac{1}{r\sin\theta}, 0\right) \\ \bar{\lambda}_{(4)^i} = \left(0, 0, 0, \frac{1}{r}\right) \end{cases}$$

The physical components $R_{(abcd)}$ of curvature tensor defined by

(3.42) $R_{(abcd)} = \lambda_{(a)^h}\lambda_{(b)^i}\lambda_{(c)^j}\lambda_{(d)^k}R_{hijk}$

are

(3.43) $R_{(2424)} = -R_{(2323)} = R_{(3434)} = \frac{1}{2r^2}$

which shows that

$$R_{(abcd)} \longrightarrow 0 \text{ as } r \to \infty$$

Hence the space time is asymptotically homaloidal.

For the metric 3.39 the fluid velocity v^i is found as

(3.44) $\upsilon^1 = \upsilon^2 = \upsilon^3 = 0 = \upsilon_1 = \upsilon_2 = \upsilon_3, \upsilon^4 = \frac{1}{r}, \upsilon_4 = r$

The scalar of expansion $\theta = u^i{}_{,i}$ is identically zero.

The non-zero components of rotation ω_{ij} and shear tensor σ_{ij} are found to be

(3.45) $\omega_{14} = -\omega_{41} = -\frac{1}{2}$

(3.46) $\sigma_{14} = -\sigma_{41} = \frac{1}{2}$

4. Solution for the Perfect fluid core

Pressure and density for the metric (3.19) are

(4.1) $$8\pi p = 8\pi\rho = \frac{12}{5r^2}\left[\frac{10}{17} + \frac{k}{r^{17/5}}\right] - \frac{1}{r^2}$$

If we consider k = 0, then equation (4.1) reduces to

(4.2) $$8\pi p = 8\pi\rho = \frac{7}{17r^2}$$

It follows from (4.1) and (4.2) that the density of the distribution tends to infinity as r tends to zero. In order to get rid of singularity at r = 0 in the density we visualize that the distribution has a core of radius r_0 and constant ρ_0. The field inside the core is given by Schwarzschild internal solution.

(4.3a) $$e^{-\alpha} = 1 - \frac{r^2}{R_0^2}$$

(4.3b) $$e^{\beta} = \left[\zeta - D\left(1 - \frac{r^2}{R_0^2}\right)\right]^2$$

(4.3c) $$8\pi p = \frac{1}{R^2}\left[\frac{3D\left(1 - \frac{r^2}{R_0^2}\right) - \zeta}{\zeta - D\left(1 - \frac{r^2}{R_0^2}\right)^{\frac{1}{2}}}\right]$$

where ζ, D are constants and $R_0^2 = \frac{3}{8\pi\rho_0}$

The continuity condition for the metric (3.19) and (4.3a – 4.3c) at the boundary gives

(4.4) $$R_0^2 = \frac{r_0^2}{\left(\frac{7}{17} - \frac{k}{r_0^{17/5}}\right)}$$

(4.5) $$\zeta = r_0^{7/10} + \frac{7R^2}{10r_0^{13/10}}\left(1 - \frac{r_0^2}{R^2}\right)$$

(4.6) $$D = \frac{7R^2}{10r_0^{13/10}}\left(1 - \frac{r_0^2}{R^2}\right)^{1/2}$$

(4.7) $K = r_0^{17/5}\left(\frac{7}{17} - \frac{r_0^2}{R^2}\right)$

and the density of the core

(4.8) $\rho_0 = \frac{3}{8\pi r^2}\left(\frac{7}{17} - \frac{k}{r_0^{17/5}}\right)$

Which complete the solution for the perfect fluid core of radius r_0 surrounded by considered fluid. The energy condition $T_{ij}u^1u_j > 0$ and the

Hawking and Penrose condition (Hawking and Penrose, 1970)

$$\left(T_{ij} - \frac{1}{2}g_{ij}T\right)u^iu_j > 0$$

Both reduces to $\rho > 0$, which is obviously satisfied.

For different value of n, solution obtained above in case I and case II provide many previously known solutions. For n = 2 and by suitable adjustment of constant we get the solution due to Singh and Yadav [17] and Yadav and Saini [25].

5. Remarks and Conclusion

In this chapter we have obtained some exact static spherical solution of Einstein's field equation with equation of state p = ρ. Our assumption is $e^{\beta} = A\, r^{\frac{an^2+bn+3}{5}}$, from which we can find value of e^{α}. We have found rotation, shear tensor, scalar of expansion. Here We have taken p = ρ, which describes several important cases, e.g. radiation, relativistic degenerate Fermi gas and probably very dense baryon matter (Zeldovich and Noviko [30] and Waleckco [21]. The casual limit for ideal gas has also the form p = ρ (Zeldovich and Novikov [30]

Further if the fluid satisfies the equation of state p = ρ and if in addition its motion is irrotational, then such a source has the same stress energy tensor as that of a massless scalar source (Tabensky and Taub [19]). Also the solution in this case can be transformed to Brons-Dicke theory in vacuum [6].

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