International Research Journal of Mathematics, Engineering and IT
ISSN: (2349-0322)
Association of Academic Researchers and Faculties (AARF)
Impact Factor- 8.132 Volume 11, Issue 09, Sep 2024
Website- www.aarf.asia, Email : editoraarf@gmail.com

QUADRATIC OPERATORS ON A LINEAR SPACE

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Abstract:
We have defined Quadratic operators on a linear space and also we extend this concept to Hilbert Space. We have studied some interesting theorems on a Quadratic operators which is in connection of spectral family.

Keywords: Linear space, Quadratic operators, Hilbert space, Self-Adjoint Operator, Positive Operator, Spectral family, Projection.

INTRODUCTION

Quadratic Operator: Let $T:H \rightarrow H$ be a positive self-adjoint operator on a complex Hilbert Space H. Then a bounded self-adjoint operator A is called a Quadratic Operator of T if

$$A^2 = T$$

. . . . (i)

In in addition, $A \geq O$ then A is called a positive quadratic operator of T and is denoted by

$A = T^{1/2}$

where, $A = T^{1/2}$ exists and is unique.[1]

Positive Operator : If T is self-adjoint (Tx, x) is real then we may consider the set of all bounded self-adjoint linear operators on a complex Hilbert space H and introduce this set a paytial ordering $\leq$ by defining

(i) $T_1 \leq T_2$, if and only if $(T_1 x, x) < (T_2 x, x)$

For all $x \epsilon H$. Instead of $T_1 \geq T_2$ we also write $T_2 \geq T_1$

A bounded self-adjoint linear operator $T:H\rightarrow H$ is said to Positive

$T \geq O$ if and only if $(Tx, X) \geq O$

For all $x \epsilon H$. Instead of $T \geq O$, we also write $O \leq T$. in fact such an operator should be called "non-negative" but "positive" is the as usual term.

The simple relation between (i) and (ii), namely $T_1 \leq T^2 \Leftrightarrow O \leq T_2 - T_1$

i.e. (i) holds if and only if $T_2 - T_1$ is positive.[2]

Role of Quadratic Operator with Spectral Family :

Application of Quadratic Operator will be considered in a Spectral family of a bounded self-adjoint linear operator. Quadratic operator will play a basic role in connection with the Spectral representation of bounded self-adjoint linear operators.[3]

Spectral Family of a Bounded Self-adjoint linear operator :

With a given self-adjoint operator $T : H \rightarrow H$ on a complex Hilbert space H we can associate a spectral family ξ such that ξ may be used for spectral representation of T.

To define ξ, we need the operator :-

1. $T_\lambda = T - \lambda I$. The positive quadratic operator of T_λ^2 which we denote B_λ, thus
2. $B_\lambda = (T_2^2)^{1/2}$ and the operator
3. $T_2^+ = 1/2\ (B_\lambda + T_\lambda)$ which is called the positive part of T_λ.

The spectral family ξ of T is then defined by $\xi = (E_\lambda)_{\lambda \in R}$, where E_λ is the projection of H onto the null space $N(T_\lambda^+)$ of T_λ^+ consider at first the operators:

$B = (T^2)^{1/2}$ (Positive part of T^2)

$T^+ = 1/2\ (B+T)$ (Positive part of T)

$T^- = 1/2\ (B-T)$ (negative part of T)

And the projections of H onto the null space of T^+ which we denote by $\in$, that is $E:H \rightarrow Y = N(T^+)$ (3)

4. $T = T^+ - T^-$

5. $B = T^+ + T^-$.[4]

THEOREMS AND LEMMAS

Theorem 1 : If two bounded self-adjoint linear operators S and T on a Hilbert Space H are positive and cominute ($ST = T_s$), then their Positve St is positive.

Proof : We show that $(S\,Tx, x) \geq O$ for all x EH. If S = O, this holds. Let S # O. We proceed in two steps :

(a) We consider, $S_1 = \frac{1}{||S||} S$, $S_{n+1} = S_n - S^2_n$

. . . . (ii)

where n - 1, 2, ... And prove by induction that $O \leq S_n \leq 1$.

. . . . (iii)

(b) We prove that $(ST_{x,\,x}) \geq O$ for all X EH.

The following are the details :

(a) For n = 1 the inequality (iii) holds. Indeed, the assumptions $O \leq S$ implies $O \leq S$, and $S_1 \leq 1$ is obtained by an application of the Schwarz inequality and the inequality

$$\| S x \| \leq \| S \| \| X \| : (S_1 x, x) = \frac{1}{||S||} (Sx, x) \leq \frac{1}{||S||} \|Sx\| \|x\|$$

$\leq \|x\|^2 = (1x, x)$

Suppose (iii) holds for an n = k, that is,

$O \leq S_k \leq 1$, thus $O \leq 1 - S_k \leq 1$

Since S_k is self-adjoint, for every x ∈ H and y = S_k X, we obtain

$(S_k^2 (1 - S_k)x, x) - ((1 - S_k) S_k x, S_k x) = ((1-S_k) y, y) \geq O.$

By definition this proves

$S_k^2 (1 - S_k) \geq O$

Similary,

$S_k (1-S_k)^2 \geq O$

By addition and simplification,

$O \leq S_k^2 (1-S_k) + (S_k (1-S_k)^2 = S_k S_k^2 = S_k + 1$

Hence, $O \leq S_{k+1}$.And $S_{k+1} \leq 1$ follows from $S_k^2 \geq O$ and $1-S_k \geq O$

by addition; indeed,

$O \leq 1 - S_k + S_k^2 = 1 - S_{k+1}.$

This completes the inductive proof of (iii).

(b) We now show that (ST x, x) $\geq$ O for all x ∈ H. From (ii)

$S_1 = S_1^2 + S_2^2 + S_3^2$

. . . .

$= S_1^2 + S_2^2 + S_3^3 + \ldots\ldots + Sn^2 + Sn+1$

Since $S_{n+1} \geq O$, this implies

$S_1^2 + S_2^2 + S_3^3 + \ldots\ldots + Sn^2 = S_1 - S_{n+1} \leq S_1.$ (iv)

By the definition of $\leq$ and the self-adjointeness of S_i this means

$\sum_{i-1}^{n} ||S_1 x||^2 = = \sum_{i=1}^{n} (S_I x, S_i x) = \sum_{1=1}^{n} (S_i^2 x, x) \leq (S_1 x, x).$

Since n is arbitary, the infinite series

$\|S_1 x\|^2 + \|S_2 x\|^2 + \ldots\ldots$ Converges.

Hence, $\| Sn x \| \to O$ and $Sn x \to$ by (iv),

$(\sum_{i=1}^{n} S_i^2) x = (S_1 - S_{n+1}) x \to S_1 x \ (n \to x)$ …. (v)

All the S_i s commute with T since they are sums and the products of $S_1 = \|S\|^{-1}$ S, and T commute.

Using S - $\|S\| S_1$, from (V), T $\geq$ O and continuity of inner product. We thus obtain for every ∈ H and Yi = Si x,

$(ST_{x, x}) = \|S\| (T S_1 x, x)$

$= \|S\| \lim_{n\to\infty} \sum_{1-1}^{n} TS_1^2 x, x)$

$= \|S\| \lim_{n\to\infty} \sum_{1-1}^{n} (Tyi, Y1) \geq O$

That is, (STx, x) $\geq$ O ……… (vi)

The partial oder relation defined by (2) also suggests the following concepts: so that Sn $Sm \geq S_n^2$, Together

$S_m^2 \geq S_n S_m \geq S_n^2$ (m < n)

By definition, using self adjointeness of Sn, we thus have

$(Sm^2\ x, x) \geq (S_n S_m\ x, x) \geq (S_n^2\ x, x) = (S_n\ x, S_{nx})$ ……….. (vii)

This shows that $(Sn^2\ x, x)$ with fixed x is a monotone decreasing sequence of non- negative numbers. Hence it converges.

We know that (Tn, x) converges. By assumption, every Tn commutes with every Tm & with K. Hence the Si ' s all commute. These operators are self adjoints.

Since $- 2 (S_n\ x, x) \leq - 2 (Sn^2\ x, x)$ by

(vii) where m < n, we thus obtain

$$\| Sm^x - Sn^x \|^2 = \{(Sm - Sn)\ x, (Sm - Sn)\ x \}$$

$$= \{((Sn - Sn)^2\ x, x \}$$

$$= (Sn^2\ x, x) - 2\ (Sm\ Sn\ x, x) + (Sn^2\ x, x)$$

$$\leq (Sm^2\ x, x) - (Sn^2\ x, x).$$

From this and the convergence proved in part (a) we see that (Snx) is cauchy. It converges since H is complete. Now Tn = k - Sn. Hence (Sn x) also converges. Clearly the limit depends on x, so that we can write Tn x $\to$ Tx for every x En hence this defines an operator T:H $\to$ H which is linear. T is self-adjoint because Tn is self-adjoint and the inner product is continuous. Since (Tn x) converges, it is bounded for every x EH.

Theorem 2: every positive bounded self-adjoint linear operator T:H $\to$ H on a complex Hilbert space H has a positive quadratic operator A, which is unique. This operator A commutes with every bounded linear operator on H which commutes with T.

Proof: We proceed in three steps :

(a) We show that if the theorem holds under the additional assumptions T $\leq$ 1, it also holds without that assumptions.

(b) We obtain the existence of operator $A = T^{1/2}$ from AnX$\to$Ax, where Ao = O and $A_{n+1} = An + {}^{1/2} (T - An^2)$, n = 0, 1, …

(c) We prove uniqueness of the positive quadratic operators.

Now,

(a) If T = O, we can take

$A = T^{1/2} = O$, Let T # O. By the Schwarz inequality.

Dividing by ||T|| # O and setting Q $\leq$ 1. Assuming that Q has a uniques positive quadratic operator B = Q $^{1/2}$, we have B^2 = Q and we see that a quadratic of T = ||T|| Q is $||T||^{1/2}$ B because $(||T||^{1/2} B)^2 = ||T|| B^2 = ||T|| Q = T$.

Also, it is not difficult to see that uniqueness of Q $^{1/2}$ implies uniqueness of the positive quadratic operator of T.

Existence : We consider (i), since A_o = O, we have A_1 = 1/2 T, $A_2 = T - 1/8 T^2$ etc. Each An is a polynomial In T. Hence the An's are self adjoints Q all commute and they also commute with every operator that T commutes with. We now prove

(ii) An $\leq$ 1, n = 0, 1, ……..

(iii) An $\leq A_{n+1}$ n = 0, 1, ……..

(iv) $A_n x \rightarrow A_x$ $A = T^{1/2}$

(v) ST = TS $\Rightarrow$ AS = SA

Where S is a bounded linear operator on H.

Proof (ii) : We have Ao $\leq$ 1. Let n > O. Since 1-A_{n-1} is self adjoint,

$(1 - A_{n-1})^2 \geq$ O. Also T $\leq$ 1 $\Rightarrow$ 1 - T $\geq$ O

From this and (i), we obtain (ii) :

O $\leq$ 1/2 (1-An-1)^2 + 1/2 (1-T)

= $1-A_{n-1} - 1/2 (T - A^2{}_{n-1})$

= $1 - A_n$

Proof (iii) : We use induction (i) gives O = Ao $\leq A_1$ = 1/2 T.

We show that $A_{n-1} \leq A_n$ for any fixed n implies

$A_n \leq A_{n-1}$.

From (i) we calculate directly

$A_{n+1} - A_n = A_n + 1/2 (T-A^2{}_n) - A_{n-1}{}^{-1/2} (T-A^2{}_{n-1})$

$= (A_n - A_{n-i}) [1 - 1/2 (A_n + A_{n+1})]$

Here $(A_n - A_{n-1}) \geq$ O by hypotehsis and [….] > O by (iii)

Jemce $A_{n+1} - A_n \geq$ O.

By product of positive operators :- If two bounded self-adjoint operators S and T on Hilbert Space H are positive and commute (ST = TS) and then their product ST is positive].[5]

Proof (iv) : $\{A_n\}$ is monotone by (iii) and $A_n \leq$ 1 by (ii).

Hence the existence of a bounded self adjoint linear operator

A such that $A_{nx} \rightarrow A_x$ for all x E H. Since $\{A_n X\}$ converges

(i) gives

$A_{n+1} X - A_n X = 1/2 (T - A^2 nx) \rightarrow O$

As $n \to \infty$. Hence $T x - A^2 x = O$ for all x, that is $T = A^2$.

Also $A \geq O$ because $O = Ao \leq An$ by (iii), that is An x,x) $\geq$ O for every x En which implies $(A_n x, x) \geq O$ for every x E H by the continuity of the inner product.

Proof (V) : We know that ST = TS implies AnS = S An, that is An S x = SA n X for all x EH.

Letting $n \to \infty$, we obtain (v)

(c) Uniqueness :

Let both A and B be positive quadratic of T.

Then $A^2 = B^2 = T$.

Also $BT = BB^2 = TB$, so that AB = BA by (V)

Let x E H be arbitrary and y = (A-B) x. Then.

$(Ay, y) \geq O$ and $(By, y) \geq O$ because $A \geq O$ and $B \geq 0$

Using AB = BA and $A^2 = B^2$, we obtain

$$
\begin{aligned}
(Ay, y) + (By, y) &= \{A+B) y, y\} \\
&= \{(A^2 - B^2) x, y\} \\
&= 0
\end{aligned}
$$

Hence (Ay, y) = (By, y) = O. Since $A \geq 0$ and A is self adjoint, it has itself positive quadratic C, that is (^{2} = A C is self adjoint. We thus obtain,

$O = (Ay, y) = (C^2 y, y) = (Cy, Cy) = ||Cy||^2$ and

Cy = O. Also $Ay = (C^2 y = C (Cy) = O$

Similarly, By = O. Hence, (A-B) Y = O. Using y = (A-B) x, we have thus for all x EH.

$||Ax - Bx||^2 = \{ (A-B)^2 x, x\} = \{(A-B) y, x \} = O$

This shows that Ax - Bx = O for all x EH and proves that A = B.

Theorem 3 : Let $T:H \to H$ be bounded positive self-adjoint linear operator on a complex Hilbert space using the positive quadratic of T, to show that for all x, y E H.

$$| (T_x, y) | \leq (Tx, x)^{1/2} (Ty, y)^{1/2}$$

also for all x E H, $|| Tx || \leq ||T||^{1/2} (Ty, y)^{1/2}$

Proof : Since $T = T^{1/2} T^{1/2}$ is self-adjoint,

$|Tx, y)| = | (T^{1/2} x, T^{1/2} y) | \leq || T^{1/2} x || \; || T^{1/2} y ||$

$= (T^{1/2} x, T^{1/2} x)^{1/2} (T^{1/2} y, T^{1/2} y)^{1/2}$

$= (Tx, x)^{1/2} (Ty, y)^{1/2}$

For all x, y, E H.

Now X E H then,

If T x = 0, then inequality holds.

Let T x # 0. We write y = Tx, we obtain,

$||Tx||^2 \leq (Tx, x)^{1/2} (T^2 x, Tx)^{1/2}$

Since,

$(T^2x , Tx) \leq ||T^2 x, T x)^{1/2}$

$\leq ||T|| \;\; ||Tx||^2$

We have

$|| Tx ||^2 \leq (Tx, x)^{1/2} \;\; || T ||^{1/2} \;\; || Tx ||$

And division by $|| Tx ||$

$|| T x || \leq (Tx, x)^{1/2} \;\; || T ||^{1/2}$

Thus quadratic operator in linear space with Hilbert space has been discussed and proved.

Lemma 4 :(operators related to T) - the operators just defined have the following properties :

(a) B, T^+ and T^- are bounded and self-adjoint.

(b) B, T^+ and T^- commute with every bounded linear operator that T commutes with; in particular.

(A) $BT = TB; T^+ T = TT^+; T^- T = TT^-; T^+ T^- = T^- T^+$

(c) $\in$ commutes with every bounded self-adjoint linear operator that T commutes with; in particular:

(B) $ET = TE$ $EB = BE$

(d) Furthermore,

(C) $T^+ T^- = 0$ $T^- T^+ = 0$

(D) $T^+ E = ET^+ = 0$ $T^- E = ET^- = T^-$

(E) $TE = - T^-$ $T (1-E) = T^+$

(F) $T^+ \geq 0$ $T^- \leq 0$

Proof :(a) is clear since T and B are bounded and self-adjoint.

(b) Suppose that TS = ST. Then $T^2S = TST = ST^2$, and BS-SB follows from theorem 2 applied to T^2

Hence,

$T + S = 1/2 \;\; (BS + TS) = 1/2 \;\; (SB + ST) = ST^+$

The proof of $T^-S = ST^-$ is similar.

(c) For every x E H we have y = Ex Ey = N (T^+).

Hence, $T^+ y = 0 \;\; ST^+ y = SO = O$. From TS = ST and (b) we have

$ST^+ = T^+ S Ex = T^+ Sy =$

$ST^+ y = 0$. Hence SE_x Ey. Since E projects H onto y.

We thus have ESE x = SE x for every X E H,

that is, ESE = SE. A Projection is self-adjoint by projection

$ES = E^* S^* = (SE)^* = (ESE)^* = E^* S^* E^* = ESE = SE.$

(d) We prove (C) - (F).

Proof of (C) :

From $B = (T^2)^{1/2}$, we have $B^2 = T^2$, Also BT = TB by (A).

Hence, again by (A),

$T^+ T^- = T^- T^+ = 1/2\ (B - T)\ 1/2(B + T)$

$= 1/4\ (B^2 + BT - TB - T^2) = 0$

Proof of (D) :

By definition, E x EN (T^+), so that $T^+ E_x = 0$ for all x E H.

Since T^+ is self adjoint, we have $ET^+ x =$

0 by (A) and (C), that is $ET^+ = T^+ E = 0$

Again $T^+ T^- x = 0$ by (C) so that T^-_x E N (T^+). Hence $ET^- x = T^- x$.

Since T^- is self-adjoint (C) yields

$T^- Ex = ET^- x = T^- x$ for all x E H, that is

$$T^- E = ET^- = T^-.$$

Proof of (E) :

From (4) by Introduction and (D) we have,

$TE = (T^+ - T^-)\ E = - T^-$ and this again by (4),

$T\ (1 - E) = T - TE = T^+$.

Proof of (F) :

By (D) and (5) of introduction and theorem (1),

$T^- = ET^- + ET^+ = E(T^- + T^+) = EB \geq 0$.

because $1 - E \geq 0$.

Lemma 5 : (Operators related to T_x) : The previous lemma remains true if we replace T, B, T^+, T^-, E by T_λ, B_λ, T_λ^+, T_λ^-, E_λ, respectively, were λ is real. Moreover, for any real K, μ, V, L the following operators all commute :

T_k, B_λ, T^+_μ, E_1.

Proof : The first statement is obvious. To obtain the second statement, we not that IS = SI and

$T_\lambda = T - \lambda I = T - \mu I + (\mu - \lambda) I = T\mu + (\mu - \lambda) I$ ------------------------------- (ix)

Hence,

$$ST = TS \Rightarrow ST\mu = T\mu S \Rightarrow ST\lambda = T\lambda S \Rightarrow SB_\lambda = B_\lambda S,\ SB\mu = B_\lambda S,$$

For $S = T_k$ that gives $T_k B_\lambda = B_\lambda T_k$, etc.

Thus we have proved that for a given bounded self-adjoint linear operator T we may define spectral family $\xi = (E_\lambda)$ in a unique fashion.

CONCLUSION

In this way we are have discussed elaborately the Quadratic operators in linear space with reference to Hilbert Space and furthermore the role of Quadratic operator with Spectral family has been presented and examined. Application of quadratic operator will be considered in Spectral family of a bounded self-adjoint linear operator. Quadratic operator will play a basic role in connection with the Spectral representation of bounded self adjoint linear operators.

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