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SOME STATIC SOLUTIONS OF EINSTEIN'S FIELD EQUAITONS FOR PERFECT FLUID SPHERE.

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ABSTRACT :

The present paper provides some exact interior solutions of the Einstein's field equations for a static spherically symmetric distribution of perfect fluid using judicious condition on metric potential. These solutions can be joined continuously to the Schwarzschild exterior solution and as such may be applicable to the investigation of stellar interiors where high central density and pressure are significant (i.e. massive bodies like non-rotating neutron stars).

Key Words : Interior solution, perfect fluid, pressure, density, metric.

1. Introduction

The well know Schwarzschild interior solution representing the field of a fluid sphere of constant density was discovered more than hundred years ago and still holds a prominent place in relativity theory. Later on Tolman [7] expressed Einstein's field equations in a mathematically convenient form and making suitable assumption on the metric co-efficients, obtained a number of interesting solutions including the solution for the Einstein universe, the Schwarzschild de-Sitter solution and the Schwarzschild interior solution.

In fact, exact solution to the Einstein's field equations in closed analytic form are difficult to obtain due to high non-linearity of the equations. So a small number of exact solutions have been obtained. The problem of constructing a static model sphere of perfect fluid (e.g. neutron model) is usually solved by numerical methods using Tolman – Oppenheimer Volkoff [3, 4, 7] equation with an equation of state specified. A small number of analytic solutions which have been obtained are valuable and interesting because one may study their properties in complete detail and with comparative case, specially their behaviour at high field intensity or high pressure and density. The analytic solutions are thus complementary to the numerical solutions obtained with reslistic equation of state.

For the case of a static spherically symmetric fluid of density ρ and pressure p, the field equations reduce to a set of three coupled ordinary differential equations involving these fluid variables and metric functions. With four unknows and three equations, the system is indeterminate. To determine the system completely we should be accompanied with one more relation. This we can achieved by specifying in some manner one of the unknows or to introduce a subsidiary relation between two of them i.e. specify an equation of state e.g. Adler [1] and Whitman [8] specified υ in such a way that the system is easily determined, where as Yadav and Saini [11] have solved the field equations by taking ρ = p. Some other workers in this line are Singh and Kumar [6], Yadav and Sharma [9] and Yadav et. al. [10].

In the present paper we have developed five solutions for the Einstein's field equations in a quite different technique by specifying the metric potential g_{11} (i.e., λ). The constants appearing in the solution have been evaluated by matching the solutions to the exterior Schwarzschild metric. Lastly we have given some properties and application of the system.

2. The Field Equations

The Einstein equations for an ideal fluid [2] are

(2.1) $$G_{\mu\upsilon} = -8\pi\left[\rho u_{\upsilon} u_{\upsilon} - (g_{\mu\upsilon} - u_{\mu} u_{\upsilon}\right]$$

where $G_{\mu\upsilon}$ is the Einstein tensor, u_{μ} is four velocity of a fluid element and $g_{\mu\upsilon}$ is the metric (throughout the investigation we set c and gravitation constants K to be unity by choice of units and specify a zero cosmological constant). For a static spherically symmetric system an appropriate metric is

(2.2) $$ds^2 = e^{\upsilon}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where λ and υ are function of r only.

Then the field equations may be written [4, 7] as

$$(2.3)\quad \frac{e^{\lambda}}{r^2} = \frac{1}{r^2} - \frac{(\upsilon^1)^2}{4} - \frac{\upsilon''}{2} + \frac{\upsilon'\lambda'}{4} + \frac{\upsilon' + \lambda'}{2r},$$

$$(2.4)\quad 8\pi p = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\upsilon'}{r}\right)$$

$$(2.5)\quad 8\pi\rho = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)$$

where a prime denotes differentiation with respect to r. We have to solve equation (2.3) for ν and λ. This may be fulfilled by quadrature in a number of ways e.g. Tolman [7] specifics various conditions on the functions ν and λ that simplify the equation and allow immediate integration while Adler [1] and Whitman [8] find λ by a judicious cnoice of *v*(r). We note that λ may be obtained if *v* is given and vice-verse.

Once ν and λ are obtained, p and ρ follow directly from equations (2.4) and (2.5). We put

$$(2.6)\quad y = e^{\upsilon/2}$$

Then using equation (2.3) we get the differential equation.

$$(2.7)\quad y'' - \left(\frac{1}{r} + \frac{\lambda'}{2}\right)y' + \left(\frac{e^{\lambda}}{r^2} - \frac{\lambda'}{2r} - \frac{1}{r^2}\right)y = 0$$

It is not always possible to get a traceable solution from the analytic specification of the equation of state. In these cases numerical and graphic techniques are easy to apply. Exact solutions in terms of known functions are most easily obtained by requiring one of the field variable to satisfy some subsidiary condition which simplify the full set of equations. Once the field equations are solved in this manner, an equation of state can be extracted. Such solution may useful in understading a system in the extreme relativistic limit, where we cannot specify a prior what the equation of state might be.

3. Solution of the field equations

As stated above the set or equations (2.3) – (2.5) cannot be solved without either choosing an equation of state or making a specific assumption an one of the functions p, ρ, λ and ν.

Case I :

We choose have

(3.1) $e^{\lambda} = \zeta$

where ζ is a constant. Now (2.7) reduces to

(3.2) $r^2 y'' - ry' + (\zeta - 1)y = 0$

On putting q + 1 = ζ and p + 1 = 0 (3.2) is transformed into

(3.3) $r^2 y'' + pry' + qy = 0$

which is well known Euler's Equation. The solution of equation (3.2) may now be written down and the metric potential $\upsilon(r)$ is obtained.

To solve equation (3.2) or (3.3) there are three possible cases [5].

Case I (a) $\zeta < 2$

The solution is

(3.4) $y = b_1 r^i + c_1 r^j, e^{\lambda} = \zeta$

Where b_1 and c_1 are constants to be fixed by the boundary conditions and

(3.5) $i = 1 + \alpha, j - 1 - \alpha \text{ for } \alpha = 2 - \zeta$

Pressure and density are given by

(3.6) $8\pi r^2 p(r) = \frac{1}{\zeta}\left[\frac{b_1(3+2\alpha)r^{2\alpha} + c_1(3-2\alpha)}{b_1 r^{2\alpha} + \zeta}\right]$

(3.7) $8\pi r^2 \rho(r) = 1 - \frac{1}{\zeta}$

The exterior metric which is extension of this present interior is necessarily the Schwarzschild metric.

(3.8) $ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$

where m is the total mess of the sphere given by

$$(3.9)\quad m = 4\pi\int_0^{r_0} \rho(r) r^2\, dr$$

where r_0 is the radius of the fluid sphere.

Continuity of first and second fundamental forms across the surface of the fluid sphere implies

$$(3.10)\quad \zeta^{-1} = \left(1 - \frac{2m}{r_0}\right)$$

(From continuity of g_{rr} using (2.2), (3.1), (3.8)

$$(3.11)\quad \left(b_1 r_0^i + c_1 r_0^j\right)^2 = \left(1 - \frac{2m}{r_0}\right)$$

(From continuity of g_{tt} using (2.2), (3.4), (3.8)

$$(3.12)\quad (b_1 r_0^i + c_1 r_0^j)(i b_1 r_0^{i-1} + j c_1 r_0^{j-1}) = \frac{m}{r_0^2}$$

(From continuity of $\frac{\partial}{\partial t}(g_{tt})$ using (2.2), (3.4), (3.8)

Equations (3.10), (3.11), (3.12) may besolved for ζ, b_1, c_1,

$$(3.13)\quad \zeta = \left(1 - \frac{2m}{r_0}\right)^{-1}$$

$$(3.14)\quad b_1 = \frac{2\alpha + \zeta - 3}{4\alpha r_0^i + \zeta^{\frac{1}{2}}}$$

$$(3.15)\quad c_1 = \frac{2\alpha - \zeta + 3}{4\alpha r_0^j + \zeta^{\frac{1}{2}}}$$

Case I (b) : $\zeta = 2$

The solution in this case is

(3.16) $y = r(b_2 \log r + c_2)$

where b_2 and c_2 are constants

pressure and density are

(3.17) $8\pi r^2 \rho(r) = \frac{1}{2},$

(3.18) $8\pi r^2 p(r) = \frac{1}{2}\left[\frac{b_2(2+3\log r)3c_2}{b_2 \log r + c_2}\right] - 1$

The constants b_2 and c_2 are given by

(3.19) $b_2 = \frac{\zeta - 3}{2r_0 \zeta^{\frac{1}{2}}}$

(3.20) $c_2 = \frac{2-(\zeta-3)\log r}{2r_0 \zeta^{\frac{1}{2}}}$

Case I(c) : $\zeta > 2$

The solution is

(3.21) $y = r^{\sigma}(b_3 \cos x + c_3 \sin x), e^{\lambda} = \zeta$

where x = δlogr and σ, δ will depend on ζ. From equations (2.4) and (2.5) pressure and density are given by

(3.22) $8\pi r^2 \rho(r) = 1 - \frac{1}{\zeta}$

(3.23) $8\pi r^2 p(r) = -1 + \frac{1}{\zeta}\left[\frac{(b_3 + 2\sigma b_3 + 2\delta c_3)\cos x + (c_3 + 2\sigma\zeta - 2\delta b_3)\sin x}{b_3 \cos x + c_3 \sin x}\right]$

where b_3 and c_3 are given by

(3.24) $b_3 = \cos x_0 \, r_0^{-\sigma} \zeta^{-\frac{1}{2}} - \frac{\sin x_0}{2r_{r_0}^{\sigma+1}\zeta^{-\frac{1}{2}}} \times \left[1 - 2(1+\sigma)\zeta^{-\frac{1}{2}}\right]$

(3.25) $c_3 = \sin x_0 r_0^{-\sigma} \zeta^{-\frac{1}{2}} + \frac{\cos x_0}{2 r_0^{\sigma+1} \zeta^{-\frac{1}{2}}} \times \zeta^{-\frac{1}{2}}$

where $x_0 = \delta \log r_0$

Case II when $\zeta = 1$

Equation (3.2) reduces to

(3.26) $r^2 y^{11} - r y^1 = 0$

whose solution is

(3.27) $y = b^1 + c^1 r^2$

where b^1 and c^1 are constants, specified by matching the solution to the exterior Schwarzschild solution at the boundary $r = r_0$

Pressure and density are given by

(3.28) $8\pi r^2 \rho(r) = 0$

(3.29) $8\pi r^2 p(r) = \frac{2c^1 r^2}{b^1 + c^1 r^2}$

Constants b^1 and c^1 are given by

(3.30) $b^1 = \frac{5-\zeta}{4\zeta^{\frac{1}{2}}}$

(3.31) $c^1 = \frac{\zeta - 1}{4 r_0^2 \zeta^{\frac{1}{2}}}$

4. Discussion and Application

For a realistic model p ≥ 0 and ρ ≥ 0 in the interior of fluid sphere. Hence in a addition these conditions will impose further restrictions on these solutions. We therefore restrict our solutions to only those values of constants for which pressure and density are positive.

The family of solutions obtained in this paper may be useful in the investigation of massive stars. They allow the investigator to very the equation of state in a continuous manner by choosing the value of parameter ζ.

If the fluid is considered adiabatic then the velocity of sound is given by the relation

$$\frac{d\rho}{dp} = -\wedge^2$$

where $\wedge$ is the speed of sound in the fluid.

5. References :

1. Adler, R.J. (1974), J. Math. Phys., **15**, 727.
2. Adler, R.J., Bazin, M. and Schiffer., M.M., Introduction to General Relativity, Mc Graw-Hill, New York, 1965, See Chap,4.
3. Misner, C., Thorne, K., Wheeler, J., Gravitation (Freeman Sanfranscisco, 1973) See Chap. 23.
4. Oppenheimer, J.R. and Volkoff, J.F. (1939) Phys. Rev. **55**, 374.
5. Ritger, P.D. and Rose. N.J., Differential Equations with Applications, Mc Graw-Hill, New York, 1968, pp. 144-146.
6. Singh, S. and Kumar M. (2019) Res. Rev. J. 4**(6)**, 1684.
7. Tolmaan, R.C. (1939) Phys. Rev. **55**, 364.
8. Whitman, P.G. (1977), J.Math. Phys. **18**, 868.
9. Yadav, A.K. and Sharma, A. (2013), Res. Astr Astrophys., 13, 501.
10. Yadav, R.B.S. et. al. (2007), Acta Ciencia Indica, 33(2), 185.
11. Yadav, R.B.S. and Saini, S.L. (1991); Astrophys. Space Science, **186**, 331-336.

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