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A CYLINDRICALLY SYMMETRIC COSMOLOGICAL MODEL IN PRESENCE OF ELECTROMAGNETIC FIELD.

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ABSTRACT :

In the present paper taking Cylindrically Symmetric metric of Marder, we have constructed a non-static spatically homogeneous Petrov type I cosmological model assuming the energy momentum tensor to be that of a perfect fluid with an electromagnetic field and the four current is either zero or space like. Various physical and geometrical properties e.g. pressure, density rotation, expansion and shear tensor have been found and discussed. We have also discussed Doppler effect and Newtonian analogue of force in the model.

Key Words : Non-static, space-like, shear, expansion, rotation.

1. Introduction :

In recent years there has been a lot of interest in cosmological models in the presence of electromagnetic fields in general relativity. A cosmological model in the presence of magnetic field has been studied by Zeldovich [23] and later by Thorne [18]. Shikin [16] also constructed a uniform axially symmetric solution (model) of Einstein-Maxwell equations in the case of propagation by an ideal fluid in the presence of magnetic field directed along the axis of symmetry. Magnetic fields in stellar bodies was also discussed by Monoghan [11]. Gravitationaly collapse of a magnetic star was studied by Greenburg [6]. Jacobs [8, 9] has studied the behaviour of the general Bianchi-type I cosmological model in the presence of a spatially homogeneous magnetic field. This problem has been studied again by De [5] with a different approach. This work has been further extended by Tupper [20] to include Einstein-Maxwell fields in which the electric field in non-zero. He has also interpreted certain type VI_{-0} cosmologies with electromagnetic field (Tupper [21]).

Roy and Prakash [14], taking the cylindrically symmetric metric of Marder[10 (a)], have constructed a spatially homogeneous cosmological model in the presence of an incident magnetic field which is also anisotropic and non-degenerate Petrov type – I. Singh and Yadav [17] constructed a spatially homogeneous cosmological model assuming the energy momentum tensor to be that of a perfect fluid with an electromagnetic field. Some other workers in this like are Bali et. al. [1, 2] Berman [3], Bhar [4], Kumar [9 (a)], Rajesh and Singh [15] Yadav and Purushttom [22], Rendal [13].

In this paper considering the cylindrically symmetric metric, we have also constructed a spatially homogeneous cosmological model assuming the energy momentum tensor to be that of a perfect fluid with an electromagnetic field and the four current as either zero or space like. Various physical and geometrical properties e.g., pressure, density, rotation, scalar of expansion and components of shear tensor have been found. We have also discussed Doppler effect and a Newtonian anologue of force in the model. When the cosmological constant $\wedge = 0$, it is found that in the absence of electromagnetic field pressure and density become equal (i.e. stiff matter) and conversely if pressure and density are equal, there is no electromagnetic field.

2. SOLUTION OF THE FIELD EQUATIONS :

We consider the most general cylindrically symmetric space time in the form given by (Marder (10(a))

(2.1) $$ds^2 = A^2(dt^2 - dx^2) - B^2 dy^2 - C^2 dZ^2$$

where the metric potentials A, B, C are functions of time t alone. This ensures that the model is spatially homogeneous.

The distribution consists of a perfect fluid and an electromagnetic field. The energy momentum tensor of the composite field is assumed to be the sum of the corresponding energy momentum tensors. Thus

(2.2) $$R_{ij} = \frac{1}{2} g_{ij} R + \wedge g_{ij} = -k[(\rho + p)u_i u_j - p g_{ij} + E_{ij}]$$

(2.3) $$g_{ij} u^i u^j = 1$$

(2.4) $$E_{ij} = g^{kl}F_{ik}F_{j1} - \frac{1}{4}g_{ij}F_{mn}F^{mn}$$

(2.5) $$F_{[ij;k]} = 0$$

(2.6) $$F^{ij}{}_{;j} = j^i$$

where E_{ij} is the electromagnetic energy momentum tensor, F_{ij} is the electromagnetic field tensor, $\wedge$ is cosmological constant, J^i is current four vector and □ and p are respectively, the density and pressure of the distribution. The co-ordinates are chosen to be comoving so that

(2.7) $$u^1 = u^2 = u^3 = 0, u^4 = \frac{1}{A}$$

We label the coordinates $(x, y, z, t) \equiv (x^1, x^2, x^3, x^4)$,

The off-diagonal components of (2.2) are

(2.8a) $$F_{12}F_{24}B^{-2} + F_{13}F_{34}C^{-2} = 0$$

(2.8b) $$F_{12}F_{14}A^{-2} - F_{23}F_{34}C^{-2} = 0$$

(2.8c) $$F_{13}F_{14}A^{-2} + F_{23}F_{24}B^{-2} = 0$$

(2.8d) $$F_{14}F_{24}A^{-2} - F_{13}F_{23}C^{-2} = 0$$

(2.8e) $$F_{14}F_{34}A^{-2} + F_{12}F_{23}B^{-2} = 0$$

(2.8f) $$F_{24}F_{34} - F_{12}F_{13} = 0$$

which lead to three possible cases :

(i) $F_{24} = F_{34} = F_{12} = F_{13} = 0$ at least one of F_{14}, F_{23} non-zero i.e., when the field F_{ij} is in x-direction only.

(ii) $F_{14} = F_{34} = F_{12} = F_{23} = 0$ at least one of F_{24}, F_{13} non-zero i.e., when the field is in y-direction only.

(iii) $F_{14} = F_{24} = F_{13} = F_{23} = 0$ at least one of F_{34}, F_{12} non-zero i.e., when the field is in z-direction only.

Hence the electromagnetic field is non-null and consists of an electric and/or magnetic field both of which are in the direction of same space axis. Without loss of generality, we may consider only case (i) in which the electric and magnetic fields are in the x-direction. We write

(2.9) $F_{14}^2 A^{-4} + F_{23}^2 B^{-2} C^{-2} = L^2$

The diagonal components of the equation (2.2) may be written as

(2.10) $\frac{2}{A^2}\left[\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{A_4 C_4}{AC} - \frac{A_4 B_4}{AB} - \frac{A_4^2}{A^2}\right] - 2\wedge$

$= -K[L^2 + (\rho + 3p)]$

(2.11) $\frac{2}{A^2}\left[\frac{A_{44}}{A} + \frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} - \frac{A_4^2}{A^2}\right] + 2\wedge$

$= K[-L^2 + (\rho - p)]$

(2.12) $-\frac{2}{A^2}\left[\frac{B_{44}}{B} + \frac{B_4 C_4}{BC}\right] + 2\wedge = -K[L^2 + (\rho - p)]$

(2.13) $-\frac{2}{A^2}\left[\frac{C_{44}}{C} + \frac{B_4 C_4}{BC}\right] + 2\wedge = -K[L^2 + (\rho - p)]$

where the suffix 4 indicates the ordinary differentiation with respect to time t after the symbols A, B, C. From these equations it is clear that L^2, □, p are each functions of time t alone.

From equations (2.5) and (2.9) it follows that F_{23} is a constant and F_{14} is a function of time t only i.e.

(2.14) $F_{23} = k, F_{14} = \pm A^2 (L^2 - k^2 B^{-2} C^{-2})^{1/2}$

where k is a constant. In the case when $F_{14} = 0$ which implies $J^i = 0$, we get the model due to Roy and Prakash [14]. Here we assume that $F_{14} \neq 0$ and find the only non-vanishing components of J^i to be

(2.15) $J^1 = \pm \frac{1}{A^2 BC} \frac{\partial}{\partial t}[BC(L^2 - k^2 B^{-2} C^{-2})^{1/2}]$

Equation (2.15) shows that J^i is space – like, unless $L^2 = l\ B^{-2} C^{-2}$

where λ is a constant in which case $J^i = 0$

The four –current J^i is in general the sum of the convection current and conduction current (Licknerowicz) [10] and Creenberg [6]

(2.16) $J^i = \in_0 u^i + \zeta u_j F^{ij}$

where $\in_0$ is the rest charge density and ζ is the conductivity. In the case considered here we have $\in_0 = 0$ i.e., magnetohydrodynamics.

From equations (2.14), (2.15) and (2.15) we find that the conductivity is given by

(2.17) $\zeta = -\frac{1}{A} D_4 D^{-1}$

where $D = BC(L^2 - k^2 B^{-2} C^{-2})^{1/2}$

The requirement of positive conductivity in (2.17) puts further restrictions on A, B, C. Hence in the magnetohydro dynamics case metric functions are restricted not only by field equations and energy conditions (Hawking and Penrose [7]), they are also restricted by the requirement that the conductivity be positive for realistic model.

The equations (2.10) – (2.13) are four equations in six unknown A, B, C, ρ, p and L. In order to determine them, two more conditions have to be imposed on them. For this we assume that the space – time is of degenerate Petrov type I, the degeneracy being in y and z directions. This requires that $C^{12}_{12} = C^{13}_{13}$. This condition is identically sstisfied if B = C. However, we shall take the metric potentials to be unequal. We further assume that F_{14} is such that

$L^2 = f^2 B^{-1} C^{-1}$ where f is a constant.

Equations (2.12) and (2.13) yield

(2.18) $\frac{B_{44}}{B} - \frac{C_{44}}{C} = 0$

Equation (2.18) with use of conditions $C^{12}_{12} = C^{13}_{13}$ yields

(2.19) $\frac{A_4}{A}\left(\frac{C_4}{C} - \frac{B_4}{B}\right) = 0$

Since $B \neq C$, equation (2.19) gives

(2.20) A = N (an arbitrary constant)

From equations (2.11), (2.12) and (2.13), we have

(2.21) $$\frac{B_{44}}{B} + \frac{B_4 C_4}{BC} = KL^2 N^2.$$

Integration of equation (2.18) gives

(2.22) $B_4 C - BC_4 = k_1 : k_1$ being an arbitrary constant of integration.

On letting $\frac{B}{C} = \alpha$ and BC = β equation (2.22) is reduced to

(2.23) $$\left(\frac{\alpha_4}{\alpha}\right)\beta = k_1$$

and equation (2.21) takes the form

(2.24) $$\frac{1}{\beta}\left[\left(\frac{\alpha_4}{\alpha} + \frac{\beta_4}{\beta}\right)\beta\right]_4 = KM^2 N^2$$

Equations (2.23) and (2.24) give

(2.25) $$\frac{\beta_{44}}{\beta} = 2KL^2 N^2$$

which after the use of condition $L^2 = f^2 B^{-1} C^{-1}$ reduces to

(2.26) $\beta_{44} = 2Kf^2 N^2$

Equation (2.26) on integrating gives

(2.7) $[\beta_4]^2 = 4 Kf^2 N^2 \beta + k_2$

where k_2 is constant of integration from (2.22) and (2.27)

We get

(2.28) $\frac{d\alpha}{\alpha} = \frac{k_1}{k_2} \frac{d\beta}{\sqrt{\frac{1}{2} - k_3}} = \frac{k_1}{k_2} \frac{\sqrt{\beta}\, d\beta}{\sqrt{1 - K_3\beta}}$

where

(2.29) $k_3 = \frac{4KN^2 f^2}{k_2}$

The transformation

(2.30) k_3 □ = y

reduces (2.28) to the form

(2.31) $\frac{d\alpha}{\alpha} = \frac{k_1}{k_2 k_3^{3/2}} \sqrt{\frac{y}{1 - y}}\, dy$

Which on integrating gives

(2.32) $\log \frac{\alpha}{k_4} = \frac{k_1}{k_2 k_3^{3/2}} \int \sqrt{\frac{y}{1 - y}} dy = k\psi(y)$ (say) = KF(□)

Where k_4, k K are constants.

Equation (2.32) may be also written as

(2.33) $\alpha = k_4 e^{KF(\beta)}$

Hence we have

(2.34) $B^2 = k_4 \beta e^{KF(\beta)}$

and

(2.35) $C^2 = \frac{\beta}{k_4} e^{-KF(\beta)}$

Therefore the metric (2.1) can be written as

(2.35) $ds^2 = N^2 \left[\frac{d\beta^2}{(d\beta / dt)^2} - dx^2 \right] - \beta^2 dy^2 - C^2 dc^2$

Which by use of equations (2.19), (2.27), (2.32) and (2.33) takes the form

$$(2.37)\quad dS^2 = N^2\left[-dx^2 + \frac{d\beta^2}{(a+P\beta)}\right] - k_4\beta e^{KF(\beta)}dy^2 - \frac{\beta}{k_4}e^{-KF(\beta)}dz^2$$

Where $P = k_2\, k_3$ and $K_2 = a$

On letting a + P□ = T, equation (2.37) may be further transformed to

$$(2.38)\quad ds^2 = N^2\left[-dx^2 + \frac{dT^2}{T}\right] - (T-a)e^{2g\phi(T)}dY^2$$

$$-(T-a)^{-1}e^{-2g\phi(T)}dZ^2$$

where g is another constant.

3. SOME PHYSICAL FEATURES

(A) The distribution in the model.

For the model (2.38) pressure p and density □ given by

$$(3.1)\quad K\rho = \frac{1}{4}(T-a)^{-2} - g^2(\phi(T))^2 + \frac{Kf^2}{2}(T-a)^{-1} - \wedge$$

$$(3.2)\quad K.p = \frac{1}{4}(T-a)^{-2} - g^2(\phi(T))^2 + \frac{3Kf^2}{2}(T-a)^{-1} - \wedge$$

The model has to satisfy the reality conditions (Ellis [5(a)]

(i) □ + p > 0,

(ii) □ + 3p > 0

which requires

$$(3.3)\quad \frac{1}{2K}(T-a)^{-2} + 2f^2(T-a)^{-1} > \frac{2g^2}{K}\{\phi'(T)\}^2$$

and

$$(3.4)\quad \frac{1}{K}(T-a)^{-2} + \frac{f^2}{2}(T-a)^{-1} + \frac{2}{K} > \frac{4g^2}{K}\{\phi'(T)\}^2$$

In the case of stiff matter (□ = p)

$$(3.5)\quad \wedge = \frac{-K}{2} f^2 (T-a)^{-1}$$

and

$$(3.6)\quad K_p = K\rho = \frac{1}{4}(T-a)^{-2} - g^2\{\phi'(T)\}^2 + Kf^2(T-a)^{-1}$$

The flow vector u□ of the distribution is given by

(3.7) $u^1 = u^2 = u^3 = 0$,

$$u^4 = \frac{1}{N}(T-a)^{3/2}\sqrt{T}$$

Tensor of rotation $W_{\mu\upsilon}$ defined by

$$(3.8)\quad W_{\mu\upsilon} = u_{\mu;\upsilon} - u_{\upsilon;\mu}$$

is identically zero. Thus the fluid filling the universe is non-rotational. The scalar of expansion $\phi = u_{\mu;\mu}$ is given by

$$(3.9)\quad \phi = \frac{1}{N}\left(T^{\mu} - a\right)^{-3/2}\sqrt{T}$$

The components of the shear tensor defined by

$$(3.10)\quad \sigma_{ij} = \frac{1}{2}(u_{i;j} + u_{j;i}) - \frac{1}{3} e \ (g_{ij} - u_i u_j)$$

are given by

$$(3.11)\quad \begin{cases} \sigma_{11} = \dfrac{N(T-a)^{-3/2}}{3}\sqrt{T} \\ \sigma_{22} = \dfrac{e^{2g\phi(T)}}{N}(T-a)^{3/2}.\left[\dfrac{1}{2} + g\phi'(T)(T-a) - \dfrac{1}{3}(T-a)^{1/2}\sqrt{T}\right] \\ \sigma_{33} = \dfrac{e^{-2g\phi(T)}}{N}(T-a)^{3/2}.\left[\dfrac{1}{2} - g\phi'(T)(T-c) - \dfrac{1}{3}(T-a)^{-1/2}\sqrt{T}\right] \\ \sigma_{44} = N(T-a)x\left[\{\dfrac{1}{2}:\dfrac{(T-a)^{3/2}}{T} - (T-c)^{1/2}\} - \dfrac{(T-a)^{-1/2}}{3\sqrt{T}}x\{T-s)T-1\}\right] \end{cases}$$

(b) The Doppler Effect in the Model

The path of light in the model (2.38) is given by

$$(3.12)\quad \left(\frac{dX}{dT}\right)^2 + (T-a)e^{2g\phi(T)}\left(\frac{dY}{dT}\right)^2 + (T-S)e^{-2g\phi(T)}\left(\frac{dZ}{dT}\right)^2 = 1$$

and for the case when the velocity is along Z-axis, equation (3.12) gives

$$(3.13)\quad \frac{dZ}{dT} = \pm N(t-a)^{-1/2}e^{g\phi(T)} = \pm\psi(T)$$

$$(3.14)\quad \int_{T_1}^{T_2}\psi(T)dT = \int_0^Z dZ$$

Hence

$$(3.15)\quad \psi_2)(T)\delta T_2 = \psi_1(T)\delta T_1 + \frac{dZ}{dT}\delta T_1 = \psi_1(T)\delta T_1 + u_z\delta T_1$$

where $\frac{dZ}{dT} = u_z$ is the Z –component of the velocity of the particle at the time of emission and $\psi_1(T)$ and $\psi_2(T)$ are the value of $\psi(T)$ for T = T_1 and T = T_2 respectively. From the above equation we get

$$(3.16)\quad \delta T_2 = \left[\frac{\psi_1(T) + u_z}{\psi_2(T)}\right]\delta T_1$$

The proper time interval □T_1 between successive wave crests as measured by the local observer moving with the source is given by

(3.17) $$\delta T_1^0 = \left\{ \frac{N^2}{T} - N^2 \left(\frac{dX}{dT} \right)^2 - (T-a)e^{2g\phi(T)} dY^2 - (T-a)e^{-2g\phi(T)} dZ^2 \right\}^{1/2} \delta T_1$$

This can be written as

(3.18) $$\delta T_1^0 = \left[\frac{N^2}{T} - u^2 \right]^{1/2} \delta T_1$$

where u is the velocity of source at the time of emission.

Similarly we may write

(3.19) $$\delta T_0^2 = N(T-a)^{1/2} T^{-1/2} \delta T_2$$

As the proper time interval between the reception of the two successive wave crests by an observer at rest at origin. Hence following Talman [19], the red shift in this case is given by

(3.20) $$\frac{\lambda + \delta\lambda}{\lambda} = \frac{\delta T_2^0}{\delta T_1^0} = \frac{(T_1 - a)^{-1/2} [N(T_1 - a)^{1/2} e^{e\phi(T)}}{[N^2 (T_2 - a)^{-1} T_2^{-2} - U^2]^{1/2} (T_2 - a)^{-1} e^{2g\phi(T)}}$$

(c) Newtonian Analogu of Force in the Model.

The vectors R_1 and S_1 are defined as follows (Narlikar and Singh, [12]).

(3.21) $$R_1 = \Delta_{j1} j = \frac{H,i}{H}$$

(3.22) $$S_1 = \Delta_{jk} 1 g^{jk} g_{1i} = g^{jk} g_{1i,k} - \frac{H,i}{H}$$

where

$$H = \sqrt{g/\gamma}$$

For the line element (2.38) we have

$$(3.23)\quad \begin{cases} g_{11} = -N^2 \\ g_{22} = -(T-a)e^{2g\phi(T)} \\ g_{33} = -(T-a)e^{-2g\phi(T)} \\ g_{44} = \dfrac{N^2}{T} \end{cases}$$

and

$$g^{11} = -\frac{1}{N^2}$$

$$g^{22} = -\frac{1}{(T-a)}e^{2g\phi(T)}$$

$$(3.24)\quad g^{33} = -\frac{1}{(T-a)}e^{2g\phi(T)}$$

$$g^{44} = \frac{T}{N^2}$$

$$(3.25)\quad g = -N^4(T-a)^2$$

The corresponding flat metric □ij is taken to be that of special relativity

(3.26) $dS^2 = -dX^2 - dY^2 - dZ^2 + dT^2$

$$(3.27)\quad y_{\mu\upsilon} = [-1,-1,-1,1]$$

and

(3.28) $y = -1$.

From equations (2.35) and (2.36) we have

$$(3.29)\quad H = \sqrt{\frac{g}{\gamma}} = N^2(T-a)$$

From (2.21) and (2.22), we get

$$(3.30)\quad R_i = [0,0,0(t-a)^{-1}]$$

(3.31) $S_i = [0, 0, 0 - (T - a)^{-1}]$

Thus we find that R_1 and S_1 both are null force vectors. R_4 and S_4 have no Newtonian analogues.

In the absence of electromagnetic field, we see

(3.32) $R_i = [0, 0, 0, \frac{1}{T}]$

and

(3.33) $S_i = [0, 0, 0, -\frac{1}{T}]$

Also pressure and density are found in this case

(3.34) $K\rho = \frac{1}{4T^2} - g^2(\phi_0'(T_1))^2 - \wedge$

(3.35) $Kp = \frac{1}{4T^2} - g^2(\phi_0'(T_1))^2 + \wedge$

where ϕ_0' is the value of ϕ' when there is no electromagnetic field.

4. CONCLUSION

When the cosmological constant $\wedge = 0$, we see that in the absence of electromagnetic field, pressure and density become equal and conversely if pressure and density are equal there is no electromagnetic field.

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