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Eigen-solution within specified bounds

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For some practical problems in vibration analysis, it is desirable to determine a few roots lying in a desired range, along with the associated vectors. The usual solution techniques compute only the first few roots and vectors and are inefficient for the effective solution of the present problem. This paper is concerned with the accurate and efficient computation of a few specified intermediate roots and vectors without having to compute any other; two associated procedures are described in detail in this paper. Both techniques are applicable for the efficient solution of the usual un-damped free vibration problem of structural systems.

Key words: Eigen-solution, finite elements, Sturm sequence, intermediate roots, progressive simultaneous iteration

1. Introduction

An accurate estimation of vibration characteristics of practical problems as aircraft and aerospace craft is a vital preliminary to compute their dynamic responses and stability characteristics. Often times, these vehicles are modular in form being also interconnected in orderly fashion. To avoid distractive resonance between the modules, it is important to compartmentalize individual module frequency spectrums and hence the importance of accurate assessment of intermediate roots and vectors.

An Eigen-value problem associated with structural vibration phenomenon has the following matrix form:

$$\left(A-\omega^2 B\right)q=0 \qquad (1.1)$$

in which A is the symmetric and positive definite stiffness matrix, B being the symmetric inertia matrix, ω are the natural frequencies, q being the vectors signifying the mode shapes.

The first procedure described herein is based on a combined Sturm sequence and inverse iteration method that enables determination of selective roots and vectors within a specified Eigen-spectrum. A progressive simultaneous iteration procedure is developed as the second solution that enables computation of first few roots and vectors within a desired, specified range of Eigen-spectrum. Matrix and finite element methods are used for structural discretization (Ref.1)

2. Numerical formulation

Two eigensolution procedures are described herein for computation of a Few desired roots and vectors

A. Combined Sturm Sequence and Inverse Iteration (SS/II) method:

In this process (Refs. 2,3) , the algorithm computes the first NR number of roots after the specified shift points, in increasing order of magnitude, along with the associated vectors. The equation of free vibration of an un-dumped structure can be written as:

$$Kq + M\ddot{q} = 0 \quad \textbf{(2.1)}$$

or using $q = \phi e^{i\omega t}$

$$\left(K - \omega^2 M\right)\phi = 0$$

Or

$$\left(K - \lambda M\right)\phi = 0 \quad \textbf{(2.2)}$$

in which $\lambda = \omega^2$, ω is the natural frequency and K *and* M are stiffness and inertia matrices respectively, ϕ are modal vectors. In a shifted form Eq. (2.2) has the following expression:

$$\left[K - \lambda^s M - \left(\lambda - \lambda^s\right)M\right]\phi = 0 \quad \textbf{(2.3)}$$

Or

$$\left(\hat{K} - \hat{\lambda}M\right)\phi = 0\,, \quad (2.4)$$

where $\hat{\lambda} = \lambda - \lambda^s$, *and* $\hat{K} = K - \lambda^s M$ and the solution is sought for the first NR roots in the vicinity of λ_s in increasing values, given the specified bounds of the desired roots, λ^u, λ^s.

A Sturm sequence procedure

The Sturm sequence procedureis adopted to isolate the first NR roots within λ^u, λ^s in an increasing order. Thus, for any value of λ, the leading principal minors $p_r(\lambda)$ of $K - \lambda M$ form a Sturm sequence, in which the number of changes in signs of consequence of $p_r(\lambda)$ is equal to the number of Eigen-values of $K - \lambda M$, having algebraic values smaller than λ. This property enables isolation of individual roots. These roots are next converged into by a special inverse iteration process [2, 3] . Thus, for a typical $(r)^{th}$ root with estimated bound λ_u^r, λ_l^r and starting with unit vectors, the following computational steps are performed during $(i+1)^{th}$ iteration for determination of both $\hat{\lambda}^r$ and the associated vector ϕ^r.

Thus,

$$\hat{\lambda}_m^r = \frac{\left(\hat{\lambda}_n^r + \hat{\lambda}_l^r\right)}{2},$$

and normalize the vector computed in the last iteration

$$\hat{\phi}_i^r = N_{i+1} M \phi_i^r \quad (2.5)$$

N_{i+1} being a suitable normalizing factor. Next solve a set of equations

$$\left(\hat{K} - \hat{\lambda}_m^r M\right)\phi_{i+1}^r = \hat{\phi}_i^r \quad (2.6)$$

and obtain a new estimate of $\hat{\lambda}^r$ at the $(i+1)^{th}$ iteration step, using the Rayleigh quotient

$$\hat{\lambda}_{i+1}^r = \frac{\left(\phi_{i+1}^r\right)^T \hat{K} \phi_{i+1}^r}{\left(\phi_{i+1}^r\right)^T M \phi_{i+1}^r}. \quad (2.7)$$

The root convergence accuracy is next checked, if

$$acc = \frac{\left|\hat{\lambda}_{i+1}^{r} - \hat{\lambda}_{i}^{r}\right|}{\left|\hat{\lambda}_{i+1}^{r}\right|}$$

is less than the desired accuracy parameter EPS then solution convergence is achieved as $\hat{\lambda}^{r} = \hat{\lambda}_{i+1}^{r}$; otherwise, the procedure is continued till convergence is achieved.

This procedure enables computation of first few NR roots within specified domain along with its vectors, without having to compute any other.

B. Progressive Simultaneous Iteration (PSI) method.

The original simultaneous iteration methods (Refs. 4,5 and 6) were developed for computation of first few roots and vectors of engineering systems. Thus, with reference to Eq. (2.3), this algorithm computes NR number of roots in the vicinity of the shift point λ_s, computing them on both sides of it, along with respective vectors. This procedure starts with performing a Cholesky factorization of the $\hat{K}$ in Eq. (2.4), only once initially as

$$\hat{K} = LDL^{T}$$

in which L is the unit lower triangular and D, a diagonal matrix. Thus starting with a set of NRT random trial vectors $\hat{Q}_i$, at a typical $(i)^{th}$ iteration step solve progressively, that is involving only the un-converged roots

$$\hat{K}Q_{i+1} = M\hat{Q}_i$$

employing the backsubstitution procedure. A vector norm test is next performed us (2.8)

and q_i^j andsupposing that NR1 roots in the vicinity of λ_s have converged then if NR1 is less than NR, then perform progressive mass orthogonalization on the unconverged (NR1-NR) roots only

$$\tilde{q}_{i+1}^{k} = q_{i+1}^{k} - \sum_{l=1}^{NR1}\left[\left(q^{l}\right)^{T} M q_{i+1}^{k}\right] q^{l} - \sum_{l=NR1+1}^{k-1}\left[\left(\hat{q}_{i+1}^{l}\right)^{T} M q_{i+1}^{k}\right]\hat{q}_{i+1}^{l} \qquad (2.9)$$

$$\hat{q}_{i+1}^{k} = \frac{\tilde{q}_{i+1}^{k}}{\left[\left(\tilde{q}_{i+1}^{k}\right)^{T} M \tilde{q}_{i+1}^{k}\right]^{\frac{1}{2}}}, \quad k = NR1+1, NRT \qquad (2.10)$$

The iteration is continued involving un-converged roots

$$\hat{Q}_{i+1} = \left[\hat{q}_{i+1}^{NR1+1}\ \hat{q}_{i+1}^{NR1+2}\ .....\hat{q}_{i+1}^{NRT}\right] \qquad (2.11)$$

till convergence is achieved; Eqs. (2.8-2.11) constitute phase one of the solution process.

In phase two, the Rayleigh-Ritz method is employed with only un-converged roots, for the further refinement of the solution [7].

3. Numerical Examples

A cantilever beam (Fig. 1) vibration problem is solved for the NR=20 roots lying within bounds PU=1500 and PL=300, along with the associated vectors. The beam is discretized with 10 elements and the SS/II technique was used for the solution. Table 1 shows the solution results for also the PU=1500 and PL=0 case, for comparison purposes.

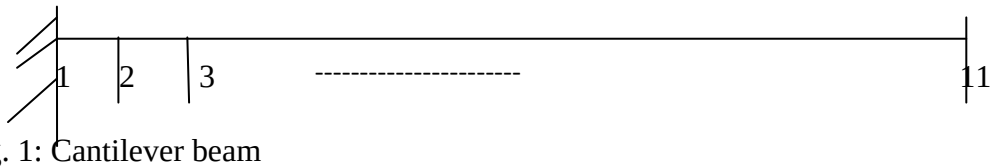

Fig. 1: Cantilever beam

Table 1: Eigen solution of a cantilever beam using the SS/II method

Mode	PU/PL	PU/PL
	1500/0.00	15.00/300.00
1	2.66	
2	3.76	In increasing
3	16.50	Increasing
4	23.33	order
5	45.73	
6	64.68	
7	88.69	↓
8	125.45	
9	144.97	
10	205.01	
11	213.66	
12	292.45	
13	302.14	302.16
14	350.88	350.83
15	375.84	375.84
16	413.60	413.53
17	452.76	452.76
18	507.44	507.44
19	531.56	531.52
20	640.30	639.62

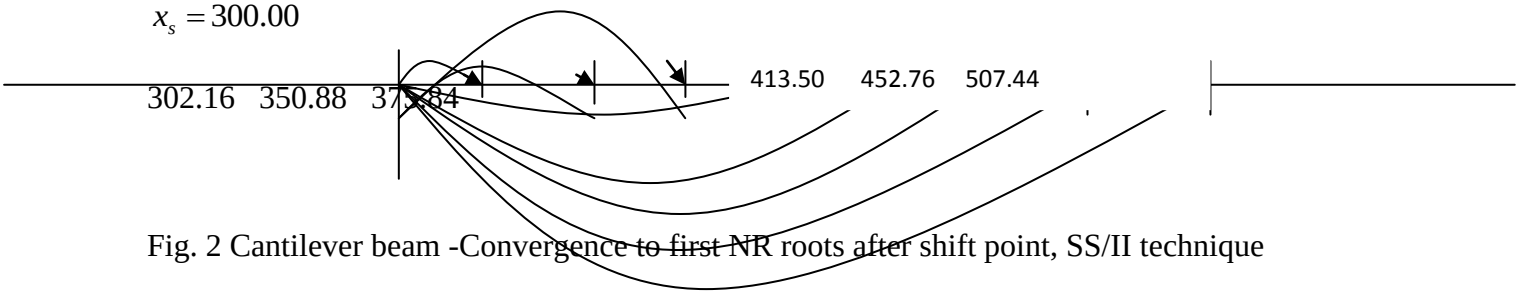

Fig. 2 Cantilever beam -Convergence to first NR roots after shift point, SS/II technique

Table 2: Cantilever beam Eigen solution using PSI method (shift λ_s=300.00)

Mode	PU/PL	PU/PL
	1500/0.00	$\lambda_s = 300$
1	2.66	
2	3.76	
3	16.50	
4	23.33	convergence
5	45.74	on
6	64.68	both sides
7	88.69	and nearest
8	125.43	to λ_s ,in no
9	144.98	particular

10	265.04	order
11	213.66	↓
12	292.45	
13	302.16	302.16
14	350.88	292.45
15	375.84	350.88
16	413.59	213.66
17	457.26	205.04
18	507.46	375.84
19	531.52	144.98
20	640.30	129.43

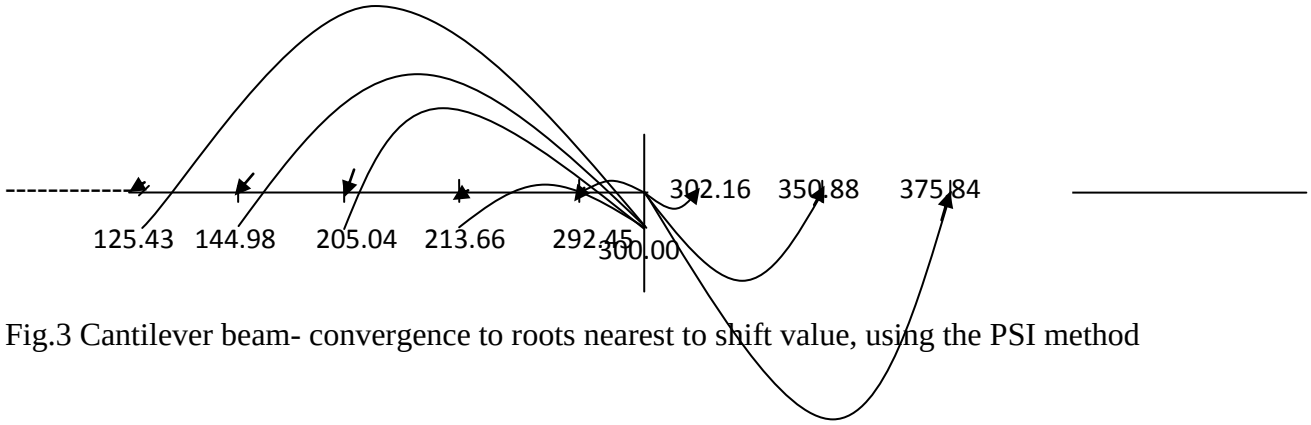

Fig.3 Cantilever beam- convergence to roots nearest to shift value, using the PSI method

Numerical Example- A generic Hypersonic Vehicle (GHV):

A more substantial example problem of vibration analysis of a complete generic hypersonic vehicle, used throughout industry for comparing solution methods, is next solved by the two techniques. The structural model consists of about 2800 nodes and 5000 elements. Table 3 shows the PSI solution results for the first few modes lying between limits PU=1500, PL=0.0 and importantly few such roots around a shift point PS=50.00, withouthaving to compute any other. Similar results using the SS/II technique are shown in Table 4. Figs.5 and 6 show interesting features of modal convergence pattern. A few typical mode shapes are shown in Fig. 7.

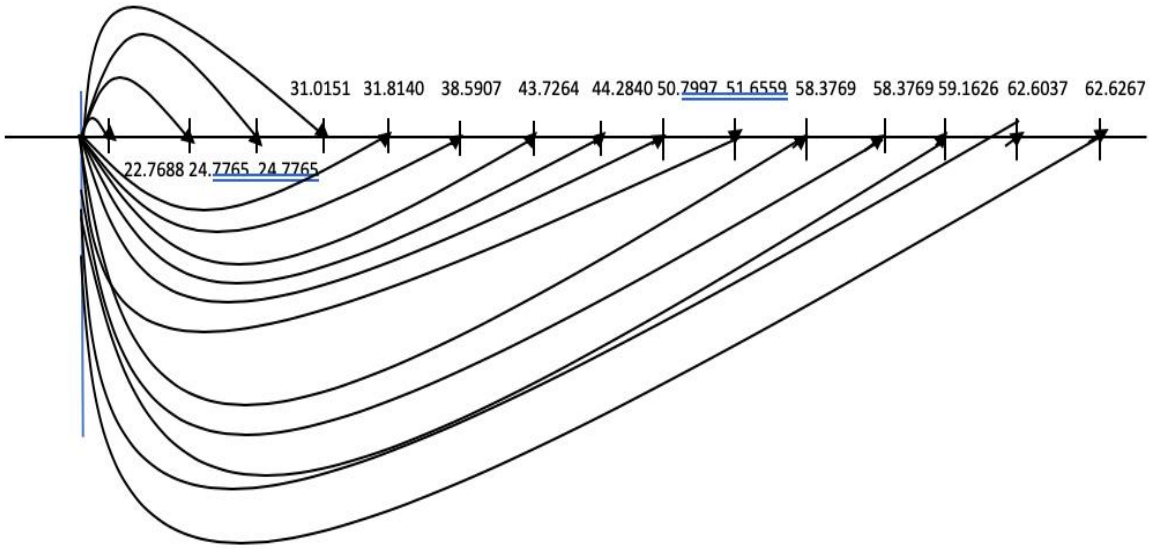

Fig. 5 Root convergence sequence of GHV using SS/II technique

Table 3: Natural Frequencies of the GHV Using SS/II Method:

mode	PU/PL	PU/PL
	λ_s =20.00	90.00/00.0
1 ↓ 6		0.0000 ↓ 0.0000
7		11.3726
8		13.4029
9		15.8105
10	22.7688	22.7869
11	24.7765	24.7767
12	24.7765	25.8781
13	31.0151	31.0151
14	31.8140	31.7140
15	38.5907	38.5908
16	43.7264	43.7406
17	44.2840	44.2837
18	50.7997	50.7997
19	51.6559	51.6509
20	58.3769	58.3769
21	58.3769	59.1423

22	59.1626	59.2873
23	62.6037	62.5956
24	62.6267	62.6586

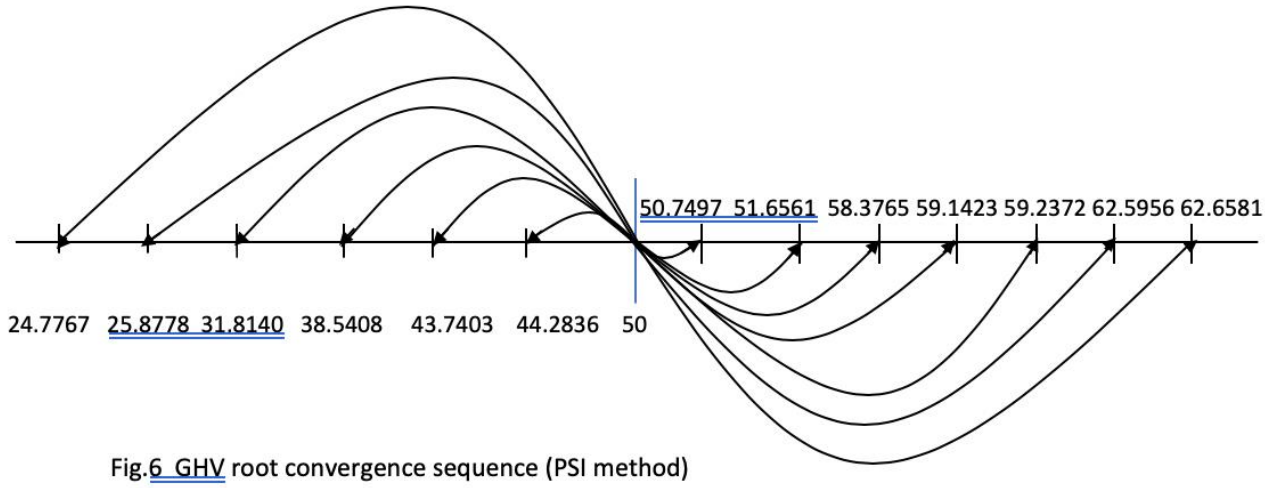

Fig.6 GHV root convergence sequence (PSI method)

Some Typical Mode Shape

Table 4: Natural Frequencies of the GHV Using PSI Method:

mode	PU/PL	PU/PL
	λ_s =1500/0.0	Shift=50.00
1	0.000	Shift=50.00
↓		
6	0.000	Convergence in no particular order, nearest to shift on both sides.
7	11.3759	
8	13.4029	
9	15.8014	
10	22.7869	
11	24.7767	
12	25.8378	
13	31.015	
14	31.814	
15	38.5907	
16	43.7403	
17	44.2840	50.797
18	50.7997	51.6561
19	51.6561	44.2836
20	58.3769	43.7408
21	59.1423	58.3765
22	59.2473	58.1423

23	62.5956	38.5968
24	62.6580	59.2872
25	67.9545	62.5950
26	68.4565	62.6581
27	77.1526	31.8140
28	77.3046	25.8778
29	80.2562	24.7767
30	82.6586	

Concluding Remarks

This paper presents two numerical algorithms based on a progressive Simultaneous Iteration(PSI) and a combined Sturm Sequence and Inverse Iteration (SS/II) method for efficient computation of intermediate roots within specified bounds and associated vectors. This is accomplished without having to compute any other and hence proves to be efficient in comparison to conventional methods that compute only the first few rots and vectors. Two numerical examples are presented that prove to the efficacy of the techniques, presented herein. In general such solutions, depending on the problems solved reduces solution time quite considerably for not having to compute the first few roots of the eigen spectrum. Also for very large problems the PSI method seems to be the more efficient one. It is interesting to observe that in the SS/II method the roots converge in increasing order on one side only whereas in th PSI method the roots converge on both sides of the shift point.

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Remarks:
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